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# Cauchy-Compact flat spacetimes with BTZ singularities

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## Abstract

The zoology of singularities for Lorentzian manifold is slightly more complex than for Riemannian manifolds. Our present work study Cauchy-compact globally hyperbolic singular flat spacetimes with extreme BTZ-like singular lines. We use the notion of BTZ-extension of a singular spacetime introduced in a previous paper to give a description of Moduli spaces of such manifolds in term of common Teichmüller spaces. This description is used to construct convex polyhedral cauchy-surface in Cauchy-compact flat spacetimes with BTZ.

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## 1 Introduction

### 1.1 Context and motivation

Let  $\mathbb{E}^{1,2}$  be Minkowski space, namely  $\mathbb{R}^3$  together with the quadratic form  $ds^2 = -dt^2 + dx^2 + dy^2$ , and let  $\text{Isom}(\mathbb{E}^{1,2}) = \text{SO}_0(1,2) \ltimes \mathbb{R}^3$  be its direct time-orientation preserving isometry group. The main object of this paper are *Cauchy-complete Cauchy-maximal globally hyperbolic singular*  $(\text{Isom}(\mathbb{E}^{1,2}), \mathbb{E}^{1,2})$ -manifolds. The most simple example of such a  $\mathbb{E}^{1,2}$ -manifold  $M$  is given by a quotient of  $F = \{-t^2 + x^2 + y^2 < 0, t > 0\}$  by a Fuchsian group, say  $\Gamma = \Gamma(2)$  the index 6 congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ . The subset  $\{-t^2 + x^2 + y^2 = -1, t > 0\}$  is a natural equivariant embedding of  $\mathbb{H}^2$  into  $F$  giving a natural *Cauchy-surface* of  $F$  and thus of  $M := F/\Gamma$ . Consider a  $\Gamma$ -invariant triangulation of  $\mathbb{H}^2$ , say  $\{\gamma T_i : \gamma \in \Gamma, i \in \{1, \dots, 6\}\}$ , we can take the suspension of each  $T_i$ ,  $\text{susp}(T_i) := (\mathbb{R}_+^* \times T_i, -dt^2 + t^2 ds_{T_i}^2)$  and glue these cones face to face to re-construct  $M$ . In this construction, the lightlike edges of the suspension have been implicitly removed. If we extend the gluing to the lightlike edges of the suspension, we obtain a  $\mathbb{E}^{1,2}$ -manifold with *extreme BTZ-like singular lines* that is a  $\mathbb{E}_0^{1,2}$ -manifold. We thus constructed a BTZ-extension [Bru16] of  $M$  say  $M'$ . It is easy to construct a polyhedral compact Cauchy-surface of  $M'$  however contrary to the natural embedding of  $\mathbb{H}^2/\Gamma$  in  $M$ , such polyhedral surface may not be convex. A good

construction of convex polyhedral Cauchy-surfaces is given by the Penner-Epstein surface Penner describes in [Pen87, Pen12]. The construction can be rephrased using the singular spacetime terminology: the boundary of the closed convex hull of a family of one point chosen on each BTZ-line is a convex polyhedral Cauchy-surface of  $M'$ .

This paper generalises these constructions to Cauchy-maximal Cauchy-compact globally hyperbolic  $\mathbb{E}^{1,2}$ -manifolds with BTZ or  $\mathbb{E}_0^{1,2}$ -manifolds following [Bru16] terminology. The main result of [Bru16] states that the regular part of such a manifold is a Cauchy-complete Cauchy-maximal globally hyperbolic  $\mathbb{E}^{1,2}$ -manifold. However, we don't *a priori* have a constant curvature Cauchy-surface in a general Cauchy-complete Cauchy-maximal  $\mathbb{E}^{1,2}$ -manifold. Therefore, a reconstruction from a triangulation of a hyperbolic Cauchy-surface is difficult to obtain and we shall proceed in a different manner. More precisely, we avoid this difficulty by constructing the maximal BTZ-extension of Cauchy-complete Cauchy-maximal  $\mathbb{E}^{1,2}$ -manifold starting from the description of its universal cover as a *regular domain* in Minkowski given by Mess, Bonsante, Benedetti and Barbot [Mes07, Bon03, Bar05, BB09].

## 1.2 Terminology and dependancies

This paper follows directly [Bru16], many elementary properties of  $(G, X)$ -structures and  $\mathbb{E}_A^{1,2}$ -manifolds that are useful to the present work can be found in this previous paper.

We use freely basic elements of the theory of  $(G, X)$ -structures in particular holonomy and developing map. See for instance section 4 of [Gol]. Most of the structures we use are special cases of  $\mathbb{E}_A^{1,2}$ -manifolds in the sense of [Bru16], i.e singular flat spacetimes. These are not  $(G, X)$ -manifolds, however they contain a regular locus i.e. a dense open subset which is a  $(\text{Isom}(\mathbb{E}^{1,2}), \mathbb{E}^{1,2})$ -manifold. The complement of the regular locus is called the singular locus. The developing map and holonomy of a  $\mathbb{E}_A^{1,2}$ -manifold are defined as the developing map and the holonomy of its regular locus. By definition, points of the singular locus are locally modeled on a singular model space. In most of the paper,  $A = \{0\}$  : the local model spaces of the singular spacetimes we are handling are  $\mathbb{E}^{1,2}$  and  $\mathbb{E}_0^{1,2}$ . Details on the geometry of the BTZ model space  $\mathbb{E}_0^{1,2}$  can be found in section 1 of [Bru16].  $\mathbb{E}_A^{1,2}$ -manifolds are singular spacetimes, the theory of (regular) semi-riemannian spacetimes is detailed in [O'N83]. Such spacetimes come with two natural orders called the time order and the causal order. Causal (resp. timelike) curves are continuous monotonic curves for the causal (resp. time) order. For a detailed exposition of properties of causal orders in semi-riemannian spacetimes see [MS08]. Many fundamental definitions and results extend to singular spacetimes [BBS11, BBS14, Bru16]. We will freely use fundamental definition and results about globally hyperbolic manifolds such as diamonds, Cauchy-surface, Cauchy-maximality, Cauchy-completeness and time functions [O'N83, Ger70, CBG69, AGH98].

A Cauchy-complete flat spacetime is absolutely maximal if it cannot be strictly embedded in any other Cauchy-complete flat spacetime. Properties of such spacetimes are given in Section 2.2 and extended to Cauchy-complete flat spacetime with BTZ in section 3.1.4.

Properties about Teichmüller theory will be given when needed. Most of the results we use can be found in [Pen12].

## 1.3 Results

The main results of the paper are :

**Theorem II.** *Let  $\Sigma$  be a closed surface and  $S$  a finite subset. There is a canonical identification between the tangent fiber bundle of the Teichmüller space of  $\Sigma \setminus S$  and the moduli space of marked Cauchy-maximal Cauchy-compact globally hyperbolic spacetimes on  $\Sigma \times \mathbb{R}$  with  $|S|$  BTZ-lines*

**Theorem IV.** *Given a Cauchy-maximal Cauchy-compact globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifolds and an arbitrary choice of points  $p_1, \dots, p_s$  in every BTZ-line  $\Delta_1, \dots, \Delta_s$ , there exists a unique convex polyhedral Cauchy-surface of vertices  $p_1, \dots, p_s$ .*

Secondary results are worth noting :

**Theorem I.** *A description of the maximal BTZ-extension of a Cauchy-maximal Cauchy-complete globally hyperbolic  $\mathbb{E}^{1,2}$ -manifold as the quotient of a convex domain of Minkowski space.*

**Theorem III.** *The moduli space of marked singular Euclidean structures on a marked surface  $(\Sigma, S)$  is canonically identified with the decorated Teichmüller space of  $(\Sigma, S)$  and the marked moduli space of linear Cauchy-compact  $\mathbb{E}_0^{1,2}$ -structures on  $(\Sigma, S)$ .*

## 2 Flat Lorentzian manifolds and Moduli spaces

### 2.1 Teichmüller space

Let  $\Sigma$  be a compact surface of genus  $g$  and let  $S$  be a finite subset of cardinal  $s \geq 0$  such that  $2g - 2 + s > 0$ . The punctured surface  $\Sigma^* := \Sigma \setminus S$  can then be endowed with a complete hyperbolic metric of finite volume. A marked surface homeomorphic to  $\Sigma^*$  is a couple  $(\Sigma_1^*, h_1)$  where  $\Sigma_1^*$  is a surface and  $h_1 : \Sigma^* \rightarrow \Sigma_1^*$  is a homeomorphism. Two marked complete hyperbolic surfaces of finite volume  $(\Sigma_1^*, h_1, m_1)$  and  $(\Sigma_2^*, h_2, m_2)$ , where  $m_i$  is a hyperbolic metric on  $\Sigma_i^*$ , homeomorphic to  $\Sigma^*$  are equivalent if there exists an isometry  $\varphi : (\Sigma_1^*, m_1) \rightarrow (\Sigma_2^*, m_2)$  such that  $h_2^{-1} \circ \varphi \circ h_1$  is a homeomorphism of  $\Sigma^*$  homotopic to the identity on  $\Sigma^*$ .

$$(\Sigma_1^*, m_1, h_1) \sim (\Sigma_2^*, m_2, h_2) \iff \begin{array}{ccc} & (\Sigma_1^*, m_1) & \\ h_1 \nearrow & \vdots & \\ \Sigma^* & \exists \varphi & \\ h_2 \searrow & \vdots & \\ & (\Sigma_2^*, m_2) & \end{array} \quad \text{with} \quad \begin{cases} h_2^{-1} \circ \varphi \circ h_1 \sim \text{Id}_{\Sigma^*} \\ \varphi \text{ isometry} \end{cases}$$

The Teichmüller space of  $(\Sigma, S)$ , denoted  $\text{Teich}_{g,s}$ , is the space of all complete hyperbolic marked surface of finite volume homeomorphic to  $\Sigma^*$  up to equivalence. Let  $\Gamma := \pi_1(\Sigma^*)$ , a point of Teichmüller space  $\text{Teich}_{g,s}$ , as all  $(G, X)$ -manifolds [Rat, Gol], has an holonomy which is a point of  $\text{Hom}(\Gamma, \text{SO}_0(1, 2))/\text{SO}_0(1, 2)$  where  $\text{SO}_0(1, 2)$  acts by conjugacy. This defines an injective map [Rat, Pen12]  $\text{Hol} : \text{Teich}_{g,s} \rightarrow \text{Hom}(\Gamma, \text{SO}_0(1, 2))/\text{SO}_0(1, 2)$ . The image is the so-called Teichmüller component [Hit92] of  $\text{Hom}(\Gamma, \text{SO}_0(1, 2))/\text{SO}_0(1, 2)$  and can be described in the following way.  $\Gamma$  has the following presentation

$$\Gamma = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s \left| \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^s c_j = 1 \right. \right\rangle.$$

The generators  $c_i$  are called peripherals and correspond to loops around the punctures  $S$ . The holonomy  $\rho$  of a point of  $\text{Teich}_{g,s}$  is marked by a choice of such generators  $(a_i, b_i, c_j)_{i \in [1, g]; j \in [1, s]}$  of  $\Gamma$ . A marked linear representation  $\rho : \Gamma \rightarrow \text{SO}_0(1, 2)$  is in the image of  $\text{Teich}_{g,s}$  by  $\text{Hol}$  if and only if it is admissible in the following sense.

**Definition 2.1** (Admissible representation (linear case)).

Let  $\Gamma = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s \mid \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^s c_j = 1 \rangle$  be a marked surface group. A marked representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(1, 2)$  is admissible if

- $\rho$  is faithful and discrete;
- for all  $j \in \{1, \dots, s\}$ ,  $\rho(c_j)$  is parabolic;
- for all  $i \in \{1, \dots, g\}$ ,  $\rho(a_i)$  and  $\rho(b_i)$  are hyperbolic.

**Remark 2.2.** This is exactly the definition of a lattice of  $\mathrm{SO}_0(1, 2)$ .

$\mathrm{Hom}(\Gamma, \mathrm{SO}_0(1, 2))/\mathrm{SO}_0(1, 2)$  has the natural compact-open topology and the Teichmüller component is a differential manifold diffeomorphic to  $\mathbb{R}^{6g-6+2s}$ . In the following we identify  $\mathrm{Teich}_{g,s}$  and the Teichmüller component of  $\mathrm{Hom}(\Gamma, \mathrm{SO}_0(1, 2))/\mathrm{SO}_0(1, 2)$ .

We now give a description of the tangent fiber bundle of  $\mathrm{Teich}_{g,s}$  following Goldman [Gol84]. Let  $[\rho] \in \mathrm{Teich}_{g,s}$  be a class of marked representation. The tangent space to  $\mathrm{Hom}(\Gamma, \mathrm{SO}_0(1, 2))$  at  $\rho$  is the space of cocycles for  $\rho$ , i.e. the space of  $\tau : \Gamma \rightarrow \mathfrak{so}(1, 2)$  such that

$$\forall \gamma_1, \gamma_2, \tau(\gamma_1 \gamma_2) = \tau(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1))\tau(\gamma_2).$$

Moreover, the action of  $\mathrm{SO}_0(1, 2)$  by conjugacy induces an equivalence of cocycles via coboundary i.e. the cocycles  $\tau : \Gamma \rightarrow \mathfrak{so}(1, 2)$  for which there exists  $u \in \mathfrak{so}(1, 2)$  such that

$$\forall \gamma \in \Gamma, \tau(\gamma) = \mathrm{Ad}(\rho(\gamma))u - u.$$

Then, for  $\rho : \Gamma \rightarrow \mathrm{SO}_0(1, 2)$  admissible, the tangent space  $T_{\mathbb{H}^2/\rho} \mathrm{Teich}_{g,s}$  is naturally identified with a subspace of  $H_{\mathrm{Ad} \circ \rho}^1(\Gamma, \mathfrak{so}(1, 2))$ .

Consider  $j \in [1, s]$  and a 1-parameter family  $(\rho_s)_{s \in \mathbb{R}}$  of admissible representations with  $\rho_0 = \rho$ . The image  $\rho_s(c_j)$  is parabolic for all  $s \in \mathbb{R}$  thus there exists a 1-parameter family  $(\phi_s)$  of elements of  $\mathrm{SO}_0(1, 2)$  such that for all  $s \in \mathbb{R}$ ,  $\rho_s(c_j) = \phi_s \rho(c_j) \phi_s^{-1}$ . A simple computation shows there exists  $u$  such that

$$\left. \frac{d\rho_s}{ds} \right|_{s=0} (c_j) = \mathrm{Ad}(\rho(c_j))u - u$$

Thus, let  $\tau$  be a tangent vector at  $\rho$ , for  $j \in [1, s]$ ,  $\tau(c_j)$  is orthogonal to the line of fixed points of  $\mathrm{Ad}(\rho(c_j))$ ; the orthogonal being taken for the Killing form on  $\mathfrak{so}(1, 2)$ . When  $s > 0$ ,  $\Gamma$  is a free group and thus dimensions are easily computed and show that the tangent vectors at  $\rho$  are exactly the cocycles satisfying this property up to coboundaries. This is still true when  $s = 0$ , see [Gol84].

Notice that the Killing form on  $\mathfrak{so}(1, 2)$  is non-degenerate of signature  $(1, 2)$  thus  $\mathfrak{so}(1, 2)$  is isometric to  $\mathbb{E}^{1,2}$ . Furthermore, adjoint action of  $\phi \in \mathrm{SO}_0(1, 2)$  on  $\mathfrak{so}(1, 2)$  is hyperbolic (resp. parabolic, resp. elliptic) if and only if  $\phi$  is hyperbolic (resp. parabolic, resp. elliptic). A point of  $T\mathrm{Teich}_{g,s}$  can then be seen as a marked representation  $\Gamma \rightarrow \mathrm{Isom}(\mathbb{E}^{1,2})$  up to conjugacy.

**Definition 2.3.** Write  $L$  the projection  $\mathrm{Isom}(\mathbb{E}^{1,2}) \rightarrow \mathrm{SO}_0(1, 2)$ . Let  $\rho : \Gamma \rightarrow \mathrm{Isom}(\mathbb{E}^{1,2})$  be a representation.

- The linear part of  $\rho$ , is  $\rho_L := L \circ \rho$ .
- The cocycle part of a  $\rho$ , is  $\tau_\rho := \rho - \rho_L$ .

For  $\phi \in \mathrm{Isom}(\mathbb{E}^{1,2})$ , we also write  $\phi_L = \phi_L$  and  $\tau_\phi := \phi - \phi_L$ .

**Definition 2.4.** For  $\phi \in \mathrm{Isom}(\mathbb{E}^{1,2})$ ,  $\mathrm{Fix}(\phi) = \{p \in \mathbb{E}^{1,2} \mid \phi x = x\}$  is a fixator of  $\phi$ .

**Definition 2.5.** Let  $\phi \in \text{Isom}(\mathbb{E}^{1,2})$ ,  $\tau_\phi$  is tangent if  $\tau_\phi \in \text{Fix}(\rho(c_j))^\perp$ .

The tangent bundle of  $\text{Teich}_{g,s}$  is then the set of admissible representations in the following sense.

**Definition 2.6** (Admissible representation (affine case)).

Let  $\Gamma = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s \mid \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^s c_j = 1 \rangle$  be a marked surface group. A marked representation  $\rho: \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  is admissible if

- $\rho_L$  is admissible;
- $\tau_\rho(c_j)$  is tangent for every  $j$ .

**Proposition 2.7.** The tangent fiber bundle of Teichmüller space  $T\text{Teich}_{g,s}$  is canonically identified with the set conjugacy classes of marked representations  $\Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$ .

## 2.2 Globally hyperbolic Cauchy-complete flat spacetimes

We briefly give main results about regular domains, a more complete study is given in [Bon03], [Bar05] and [BB09]. We use notations from [O'N83], the timelike (resp. causal) future of a set  $A$  is denoted  $I^+(A)$  (resp.  $J^+(A)$ ). Such a manifold has a natural order relation, the causal order. The map

$$\text{rev}: \begin{array}{ccc} \mathbb{E}^{1,2} & \longrightarrow & \mathbb{E}^{1,2} \\ (t, x, y) & \longmapsto & (t, x, y) \end{array}$$

is the time reversal map. This transformation preserves the quadratic form, it is in  $O(1,2)$  but not in  $SO_0(1,2)$ . The time reversal map induces an involution on the set of  $\mathbb{E}^{1,2}$ -manifolds: let  $M$  be a  $\mathbb{E}^{1,2}$ -manifold, we can replace every local chart  $\mathcal{U} \subset M$ ,  $\varphi: \mathcal{U} \rightarrow \mathbb{E}^{1,2}$  by  $\text{rev} \circ \varphi: \mathcal{U} \rightarrow \mathbb{E}^{1,2}$ . This inverse the causal order on  $M$ . This transformation is called *time reversal*.

An affine lightlike plane in  $\mathbb{E}^{1,2}$  is of the form  $\{x \in \mathbb{E}^{1,2}, \langle x|u \rangle = \lambda\}$  for some  $u$  lightlike vector and  $\lambda \in \mathbb{R}$ . The set of affine lightlike plane is then homeomorphic to  $S^1 \times \mathbb{R}$ .

**Definition 2.8** (Regular domain). A regular domain  $\Omega \subset \mathbb{E}^{1,2}$  is a set of the form

$$\Omega_\Lambda := \bigcap_{\Pi \in \Lambda} I^+(\Pi)$$

for some closed family of lightlike planes  $\Lambda$ .

**Theorem 2.1** ([Bar05]). Let  $\Gamma$  be a discrete subgroup of  $\text{Isom}(\mathbb{E}^{1,2})$  torsionfree such that  $\Gamma \cdot \Omega = \Omega$ . Then  $\Gamma$  acts properly discontinuously on  $\Omega$  and  $\Omega/\Gamma$  is a globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold.

**Theorem 2.2** ([Bar05, BB09]). Let  $M$  be a maximal globally Cauchy-complete hyperbolic space-time. Then the developping map is an embedding so that the universal covering  $\tilde{M}$  of  $M$  can be identified with a domain  $\Omega \subset \mathbb{E}^{1,2}$ . Moreover, up to time inversion, one of the following holds.

- (i)  $\Omega = \mathbb{E}^{1,2}$  and the holonomy group acts as a free abelian group of spacelike translations of rank at most 2.
- (ii)  $\Omega$  is the future of a lightlike plane  $\Pi$  and the holonomy group, if non trivial, is generated by a spacelike translation or a parabolic linear isometry.
- (iii)  $\Omega = I^+(\Pi^-) \cap I^-(\Pi^+)$ , where  $\Pi^+, \Pi^-$  are parallel lightlike planes. The holonomy group, if non trivial, is generated by a spacelike translation or a parabolic linear isometry.

(iv)  $\Omega$  is a regular domain, the linear part of the holonomy  $\pi_1(M) \rightarrow \text{SO}_0(1,2)$  is a faithful and discrete representation.

**Corollary 2.9.** *In particular a Cauchy-maximal globally hyperbolic Cauchy-complete spacetime is future complete up to time reversal.*

**Definition 2.10.** *A spacetime  $M$  is of type (i) (resp. (ii), resp. (iii), resp. (iv)) if it is a globally hyperbolic, Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold falling into case (i) (resp. (ii), resp. (iii), resp. (iv)) of Theorem 2.2*

**Definition 2.11.** *Let  $M$  be a globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold,  $\Gamma$  be its fundamental group and  $\rho$  its holonomy.  $M$  is absolutely maximal if for all globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold  $M'$  and for all injective morphism of  $\mathbb{E}^{1,2}$ -structure  $i : M \rightarrow M'$ ,  $i$  is surjective.*

**Proposition 2.12** ([Bar05]). *Let  $M$  be a globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold, then there exists an absolutely maximal globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold  $\bar{M}$  in which  $M$  embeds isometrically. Moreover,  $\bar{M}$  is unique up to isomorphism.*

**Proposition 2.13** ([Bar05]). *Let  $M$  be a globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold, let  $\Gamma := \pi_1(M)$  and let  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  be its holonomy. Define*

$$\Omega(\rho) = \bigcap_{\Pi \in \Lambda(\rho)} I^+(\Pi)$$

where  $\Lambda(\rho)$  is the closure of the set of lightlike plane which are repellent fixed point of hyperbolic element of  $\rho(\Gamma)$ .

Then the absolutely maximal extension of  $M$  is isomorphic to  $\Omega(\rho)/\rho$ .

**Lemma 2.14.** *Let  $\phi \in \text{Isom}(\mathbb{E}^{1,2})$  parabolic, the following are equivalent :*

(i)  $\text{Fix}(\phi) \neq \emptyset$

(ii)  $\tau_\phi \in \text{Fix}(\phi_L)^\perp$

*Proof.* To begin with, these two properties are invariant under conjugation of  $\phi$ . Assume  $\text{Fix}(\phi) \neq \emptyset$  and let  $p \in \mathbb{E}^{1,2}$  such that  $\phi p = p$ , up to a conjugation we can assume  $p = O$  and thus  $\tau_\phi = 0 \in \text{Fix}(\phi_L)^\perp$ .

Assume  $\tau_\phi \in \text{Fix}(\phi_L)^\perp$ . Conjugating  $\phi$  by a translation of vector  $u$  changes  $\tau_\phi$  into  $\tau_\phi + (\phi_L - 1)u$ . The map  $\mathbb{E}^{1,2} \xrightarrow{\phi_L - 1} \mathbb{E}^{1,2}$  is linear of rank 2 since  $\phi_L$  has exactly one direction of fixed points. Since  $\text{Im}(\phi_L - 1) \subset \text{Fix}(\phi_L)^\perp$  and both are of dimension 2, then  $\text{Im}(\phi_L - 1) = \text{Fix}(\phi_L)^\perp$ . Therefore, there exists  $u \in \mathbb{E}^{1,2}$  such that the conjugation of  $\phi$  by the translation of vector  $u$  is linear. Finally,  $\phi$  is conjugated to an isometry which admits a fixed point and thus  $\text{Fix}(\phi) \neq \emptyset$ .  $\square$

**Lemma 2.15.** *Let  $\Omega$  be a regular domain stabilized by some torsionfree discrete subgroup  $G \subset \text{Isom}(\mathbb{E}^{1,2})$ . Then for all  $\phi \in G$  parabolic,  $\tau_\phi$  is tangent.*

*Proof.* See [Bar05] section 7.3.  $\square$

**Corollary 2.16.** *Let  $\Sigma$  be a compact surface,  $S$  be a finite subset of  $\Sigma$  and  $\Gamma = \pi_1(\Sigma^*)$  such that  $2g - 2 + s > 0$ . Let  $M$  be a globally hyperbolic Cauchy-complete spacetime homeomorphic to  $(\Sigma^*) \times \mathbb{R}$  and  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  be its holonomy.*

*Then  $\rho$  is admissible if and only if  $\rho_L$  is admissible.*

*Proof.* If  $\rho$  is admissible, by definition,  $\rho_L$  is admissible. If  $\rho_L$  is admissible then  $\rho(\Gamma)$  fixes some non-empty regular domain and thus, by Lemma 2.15,  $\tau_{\rho(\gamma)}$  is tangent whenever  $\rho(\gamma)$  is parabolic and thus  $\rho$  is admissible.  $\square$

We end this section by a remark on the topology of Cauchy-complete globally hyperbolic flat spacetimes.

**Proposition 2.17.** *Let  $\Sigma^*$  be a punctured surface of genus  $g \geq 0$  with  $s \geq 0$  punctures such that  $2g - 2 + s > 0$  and  $\Gamma$  its fundamental group. Let  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  be a discrete faithful marked representation of  $\Gamma$ . Let  $M$  be a globally hyperbolic Cauchy-complete spacetime of fundamental group  $\pi_1(M)$  isomorphic to  $\Gamma$ .*

*If the holonomy of  $M$  is  $\rho$  then  $M$  is homeomorphic to  $\Sigma^* \times \mathbb{R}$ .*

*Sketch of proof.* We let  $\Gamma$  acts on  $\mathbb{E}^{1,2}$  via  $\rho$  and acts on  $\mathbb{H}^2$  via  $\rho_L$ . Since  $\Gamma$  is not abelian, from Theorem 2.2,  $\rho_L$  is faithful and discrete.

Write  $\Omega$  a  $\Gamma$ -invariant regular domain such that  $M$  is isomorphic to  $\Omega/\Gamma$ . From Proposition 3.3.3 in [BB09], there exists a  $\Gamma$ -invariant convex  $\mathcal{C}^1$  Cauchy-surface of  $\Omega$ , say  $\tilde{\Sigma}_1^*$  and  $\Sigma_1^* := \tilde{\Sigma}_1^*/\Gamma$  is a Cauchy-surface of  $M$ . The Gauss map of  $\tilde{\Sigma}_1^*$  defines a map  $N : \Sigma_1^* \rightarrow \mathbb{H}^2$ .  $\tilde{\Sigma}_1^*$  is the graph of a convex function defined on  $\mathbb{R}^2$  thus from a Theorem of Minty [Min61, Min64], there exists a closed convex domain  $H \subset \mathbb{H}^2$  such that  $\text{Int}(H) \subset N(\Sigma_1^*) \subset H$ . We now prove that  $N$  is proper. Sections 3.4 to 3.6 of [BB09] explain how to construct a geodesic lamination on  $H$  from  $\Omega$  and Proposition 3.6.1 shows that the inverse image by  $N$  of a compact curve  $c$  in  $H$  is of finite length if and only if it does not intersects  $\partial H$ . This implies that  $N$  is proper on  $N^{-1}(\text{Int}(H))$  and thus  $N^{-1}(\text{Int}(H))/\Gamma$  and  $\text{Int}(H)/\Gamma$  have the same number of ends. Since  $\text{Int}(H)/\Gamma$  and  $N^{-1}(\text{Int}(H))/\Gamma$  have the same finitely generated fundamental group and the same number of ends, they are homeomorphic.

A simple analysis of the way  $\tilde{\Sigma}_1^*$  is constructed shows that the inverse image of the boundary of  $H$  is a set of the form  $\gamma + a + \mathbb{R}_+ \cdot h$  for some spacelike vector  $h$ , some vector  $a$  and some  $\gamma$  geodesics of  $\mathbb{H}^2$ . Therefore,  $\Sigma_1^*$  is obtained from  $N^{-1}(\text{Int}(H))/\Gamma$  by extending its ends, it is thus homeomorphic. Finally,  $\Sigma_1^*$  is homeomorphic to  $\text{Int}(H)/\Gamma$ , which is homeomorphic to  $\Sigma^*$ .  $\square$

### 2.3 Flat Lorentzian Moduli spaces

The definition of  $\text{Teich}_{g,s}$  can be adapted to define Moduli spaces associated to marked  $\mathbb{E}_A^{1,2}$ -structures.

**Definition 2.18** (Equivalence of marked  $\mathbb{E}_A^{1,2}$ -manifolds). *Let  $\Sigma$  be a non necessarily compact surface.*

*Let  $(M_1, h_1, m_1), (M_2, h_2, m_2)$  be marked  $\mathbb{E}_A^{1,2}$ -manifolds, where  $h_i : \Sigma \times \mathbb{R} \rightarrow M_i$  is a homeomorphism and  $m_i$  is a  $\mathbb{E}_A^{1,2}$ -structure on  $M_i$ . The manifolds  $M_1$  and  $M_2$  are equivalent if there exists a  $\mathbb{E}_A^{1,2}$ -isomorphism  $\varphi : M_1 \rightarrow M_2$ , such that  $h_2^{-1} \circ \varphi \circ h_1$  is homotopic to  $(x, t) \mapsto (x, t)$  or  $(x, t) \mapsto (x, -t)$ .*

**Remark 2.19.** *As stated before, the reversal of time is a transformation among spacetimes which is not an isomorphism of  $(\text{Isom}(\mathbb{E}^{1,2}, \mathbb{E}^{1,2})$ -structure. However, the properties of a  $\mathbb{E}^{1,2}$ -manifold is closely related to the properties of its time reversal. We thus authorise time reversal as equivalence.*



**Definition 2.20** (Linear marked  $\mathbb{E}^{1,2}$ -Moduli space). *Let  $\Sigma$  be a compact surface of genus  $g$ ,  $S$  be a finite subset of cardinal  $s > 0$ . The Moduli space  $\mathcal{M}_{g,s}^L(\mathbb{E}^{1,2})$  is the space of equivalence classes of marked  $\mathbb{E}^{1,2}$ -manifolds of **linear admissible** holonomy homeomorphic to  $(\Sigma^*) \times \mathbb{R}$  which are globally hyperbolic Cauchy-complete, Cauchy-maximal.*

**Definition 2.21** (Marked  $\mathbb{E}^{1,2}$ -Moduli space). *Let  $\Sigma$  be a compact surface of genus  $g$ ,  $S$  be a finite subset of cardinal  $s > 0$ . The Moduli space  $\mathcal{M}_{g,s}(\mathbb{E}^{1,2})$  is the space of equivalence classes of marked  $\mathbb{E}^{1,2}$ -manifolds of **affine admissible** holonomy homeomorphic to  $(\Sigma^*) \times \mathbb{R}$  which are globally hyperbolic Cauchy-complete, Cauchy-maximal and **absolutely maximal**.*

**Definition 2.22** (Linear marked  $\mathbb{E}_0^{1,2}$ -Moduli space). *Let  $\Sigma$  be a compact surface of genus  $g$ ,  $S$  be a finite subset of cardinal  $s > 0$ . The Moduli space  $\mathcal{M}_{g,s}^L(\mathbb{E}_0^{1,2})$  is the space of equivalence classes of marked  $\mathbb{E}_0^{1,2}$ -manifolds of **linear** holonomy homeomorphic to  $\Sigma \times \mathbb{R}$  which are globally hyperbolic, Cauchy-maximal, and such that the marking sends  $S \times \mathbb{R}$  on  $\text{Sing}_0$ .*

**Definition 2.23** (Marked  $\mathbb{E}_0^{1,2}$ -Moduli space). *Let  $\Sigma$  be a compact surface of genus  $g$ ,  $S$  be a finite subset of cardinal  $s > 0$ . The Moduli space  $\mathcal{M}_{g,s}(\mathbb{E}_0^{1,2})$  is the space of equivalence classes of  $\mathbb{E}_0^{1,2}$ -manifolds homeomorphic to  $\Sigma \times \mathbb{R}$  which are globally hyperbolic, Cauchy-maximal, and such that the marking sends  $S \times \mathbb{R}$  on  $\text{Sing}_0$ .*

## 2.4 First correspondances between Moduli spaces

All the Moduli correspondances given in the paper are equivariant under the action of the mapping class group. This is straightforward considering the constructions are explicit, proofs are thus omitted. We use brackets,  $[M]$ , to designate the equivalence class of a manifold  $M$ . Theorem 2.1 gives a simple correspondance between  $\text{Teich}_{g,s}$  and  $\mathcal{M}_{g,s}^L(\mathbb{E}^{1,2})$  which can be given simply. Let  $\Sigma$  a surface of genus  $g$  and  $S$  a finite subset of  $\Sigma$  of cardinal  $s > 0$  with  $2g - 2 + s > 0$ . Let  $\Gamma$  be the fundamental group of  $\Sigma^*$ .

**Definition 2.24** (Suspension). *Define the suspension from  $\mathbb{H}^2$  to  $\mathbb{E}^{1,2}$*

$$\text{susp}_{\mathbb{H}^2} : \begin{cases} \text{Teich}_{g,s} & \longrightarrow \mathcal{M}_{g,s}^L(\mathbb{E}^{1,2}) \\ [\Sigma_1^*, ds^2] & \longmapsto [\mathbb{R}_+^* \times \Sigma_1^*, -dT^2 + T^2 ds^2] \end{cases} .$$

*This map comes with a natural embedding of the  $\mathbb{H}^2$ -surface into its image  $\mathbb{E}^{1,2}$ -manifold, namely the surface  $T = 1$ .*

The developpement of a point  $M$  of  $\mathcal{M}_{g,s}^L(\mathbb{E}^{1,2})$  can be chosen such that its image is the future cone  $I^+(O)$ . There is a natural time function on  $I^+(O)$ ,  $T : (t, x, y) \mapsto -t^2 + x^2 + y^2$ .

**Proposition 2.25.** *The map  $\text{susp}_{\mathbb{H}^2}$  is bijective of inverse :*

$$\text{susp}_{\mathbb{H}^2}^{-1} : \begin{cases} \mathcal{M}_{g,s}^L(\mathbb{E}^{1,2}) & \longrightarrow \text{Teich}_{g,s} \\ [M] & \longmapsto [(T \circ \mathcal{D}_M)^{-1}(1)] \end{cases}$$

*with  $T(t, x, y) = -t^2 + x^2 + y^2$  and  $\mathcal{D}_M$  a developpement of  $M$  onto  $I^+(O)$ .*

*Proof.* Let  $[M]$  be a point of  $\mathcal{M}_{g,s}^L(\mathbb{E}^{1,2})$ , since the holonomy of  $M$  is linear, the developpement of  $M$  is the future  $I^+(p)$  for some  $p \in \mathbb{E}^{1,2}$  which can be chosen to be  $O$  the origin of  $\mathbb{E}^{1,2}$ . The fundamental group  $\pi_1(M)$  then acts on  $I^+(O)$  via a lattice representation  $\rho$  in  $\text{SO}_0(1, 2)$ . The surface  $T = 1$  is an isometric embedding of  $\mathbb{H}^2$  into  $I^+(O)$  thus  $(T \circ \mathcal{D}_M)^{-1}(1)$  is a riemannian surface isometric to  $\mathbb{H}^2/\rho$  which is a point of  $\text{Teich}_{g,s}$ . The suspension of this surface is constructed by identifying its universal cover, namely  $\mathbb{H}^2$ , with the surface  $T = 1$  in  $I^+(O)$ , this shows that the suspension of  $\mathbb{H}^2/\rho$  is  $I^+(O)/\rho \simeq M$ .  $\square$

**Remark 2.26.** For all  $[\Sigma_1^*, ds^2] \in \text{Teich}_{g,s}$ , the fundamental group of the suspension  $\text{susp}_{\mathbb{H}^2}([\Sigma_1^*, ds^2])$  is canonically identified with the one of  $\Sigma_1^*$ , the suspension preserves the marking and the holonomy of  $\text{susp}_{\mathbb{H}^2}([\Sigma_1^*, ds^2])$  is the same as the holonomy of  $[\Sigma_1, ds^2]$ .

We thus obtain the following diagram :

$$\text{Teich}_{g,s} \begin{array}{c} \xrightarrow{\text{susp}_{\mathbb{H}^2}} \\ \xleftarrow{\text{susp}_{\mathbb{H}^2}^{-1}} \end{array} \mathcal{M}_{g,s}^L(\mathbb{E}^{1,2})$$

**Proposition 2.27.** Identifying  $T\text{Teich}_{g,s}$  and the classes of marked admissible representations  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$ , the holonomy defines an injective map

$$\text{Hol} : \mathcal{M}_{g,s}(\mathbb{E}^{1,2}) \rightarrow T\text{Teich}_{g,s}$$

*Proof.* Let  $[M] \in \mathcal{M}_{g,s}(\mathbb{E}^{1,2})$  and let  $\rho$  be its holonomy.  $\rho_L$  is admissible by definition of  $\mathcal{M}_{g,s}(\mathbb{E}^{1,2})$  and by Corollary 2.16 so is  $\rho$ . The injectivity follows from the construction of the absolutely maximal extension of a globally hyperbolic Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold which only depends on the holonomy.  $\square$

Remains the question of the surjectivity of  $\text{Hol}$ . From Proposition 2.13, it suffices to construct a globally hyperbolic Cauchy-complete space-time of given admissible marked holonomy. This is the object of section 3.2.

In [Bru16], the maximal BTZ-extension of a  $\mathbb{E}^{1,2}$ -manifold has been introduced as well as the regular part of a  $\mathbb{E}_0^{1,2}$ -manifold. Denote these two constructions by BTZ – ext and Reg.

**Theorem 2.3** (Theorem 2 in [Bru16]). *Let  $A \subset \mathbb{R}_+$ , let  $M$  be a globally hyperbolic  $\mathbb{E}_A^{1,2}$ -manifold. There exists a maximal BTZ-extension  $\overline{M}$  of  $M$ . Furthermore it is unique up to isometry.*

**Theorem 2.4** (Theorem 3 in [Bru16]). *Let  $M$  be a globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifold, the following are equivalent.*

- (i)  $\text{Reg}(M)$  is Cauchy-complete and Cauchy-maximal.
- (ii)  $\text{BTZ – ext}(M)$  is Cauchy-complete and Cauchy-maximal.

Consider a point  $[M]$  of the  $\mathbb{E}^{1,2}$ -moduli space, its BTZ-extension is Cauchy-complete and Cauchy-maximal. In Section 3.1, we show that  $\text{BTZ – ext}(M)$  is Cauchy-compact thus a point of the  $\mathbb{E}_0^{1,2}$ -moduli space. Consider now a point  $[N]$  of the  $\mathbb{E}_0^{1,2}$ -moduli space, its regular part is Cauchy-maximal and Cauchy-maximal. We will show in Section 3.3 that  $\text{Reg}(N)$  has admissible holonomy and is thus a point of the  $\mathbb{E}^{1,2}$ -moduli space.

$$\mathcal{M}_{g,s}(\mathbb{E}^{1,2}) \begin{array}{c} \xrightarrow{\text{BTZ-ext}} \\ \xleftarrow{\text{Reg}} \end{array} \mathcal{M}_{g,s}(\mathbb{E}_0^{1,2})$$

The constructions BTZ – ext and Reg being inverse to each other, they will automatically define bijections between moduli spaces. In order to complete the picture, we will construct a map  $\text{dsusp}_{\mathbb{H}^2} : T\text{Teich}_{g,s} \rightarrow \mathcal{M}_{g,s}^{\mathbb{E}^{1,2}}$  in Section 3.2 Finally, the following diagram sum-up the situation:

$$\begin{array}{ccccc} \text{Teich}_{g,s} & \begin{array}{c} \xrightarrow{\text{susp}_{\mathbb{H}^2}} \\ \xleftarrow{\text{susp}_{\mathbb{H}^2}^{-1}} \end{array} & \mathcal{M}_{g,s}^L(\mathbb{E}_0^{1,2}) & \begin{array}{c} \xrightarrow{\text{BTZ-ext}} \\ \xleftarrow{\text{Reg}} \end{array} & \mathcal{M}_{g,s}^L(\mathbb{E}^{1,2}) \\ \\ T\text{Teich}_{g,s} & \begin{array}{c} \xrightarrow{\text{dsusp}_{\mathbb{H}^2}} \\ \xleftarrow{\text{Hol}} \end{array} & \mathcal{M}_{g,s}(\mathbb{E}^{1,2}) & \begin{array}{c} \xrightarrow{\text{BTZ-ext}} \\ \xleftarrow{\text{Reg}} \end{array} & \mathcal{M}_{g,s}(\mathbb{E}_0^{1,2}) \end{array}$$

### 3 Spacetime constructions and Moduli spaces

#### 3.1 Maximal BTZ-extension of regular domains

We give ourselves a regular domain  $\Omega$  invariant under the action of some discrete torsionfree subgroup  $G \subset \text{Isom}(\mathbb{E}^{1,2})$ . The aim of this section is to give a simple description of the maximal BTZ-extension of  $\Omega/G$  as the quotient  $\tilde{\Omega}/G$  of some augmented domain  $\tilde{\Omega}$ . We also prove the Cauchy-compactness of the maximal BTZ-extension of an absolutely maximal Cauchy-complete globally hyperbolic  $\mathbb{E}^{1,2}$ -manifold of admissible holonomy. The main result of the section are Theorem I, which gives an explicit construction of the maximal BTZ-extension of a Cauchy-complete globally hyperbolic  $\mathbb{E}^{1,2}$ -manifold. Proposition 3.22 and Corollary 3.25 give precision on Theorem I in special cases. Also, Definitions 3.3 and 3.5 of augmented regular domain as well as Definition 3.15 of absolutely maximal spacetimes are important in most of what follows.

##### 3.1.1 Example and augmented regular domain

Recall the model space of BTZ singularities, namely  $\mathbb{E}_0^{1,2}$ , is defined by  $\mathbb{R}^3$  with the semi-riemannian metric  $ds^2 = -d\tau dr + dr^2 + r^2 d\theta$  in cylindrical coordinates. The line  $\text{Sing}_0(\mathbb{E}_0^{1,2}) = \{r = 0\}$  is singular and its regular part,  $\text{Reg}(\mathbb{E}_0^{1,2}) = \{r > 0\}$ , is a  $\mathbb{E}^{1,2}$ -manifold. A developping map  $\mathcal{D} : \text{Reg}(\mathbb{E}_0^{1,2}) \rightarrow \mathbb{E}^{1,2}$  is given in Proposition 15 and its Corollary in [Bru16]. The image of this developping map (hence every) is the chronological future of some lightlike line  $\Delta$ . Notice that  $I^+(\Delta)$  is an open half-space delimited by the lightlike plane  $\Delta^\perp$ . The holonomy is the group  $\langle \phi \rangle$  where  $\phi$  point-wise stabilizes  $\Delta$  and  $\mathcal{D}$  induces a homeomorphism

$$\overline{\mathcal{D}} : \text{Reg}(\mathbb{E}_0^{1,2}) \xrightarrow{\sim} I^+(\Delta)/\langle \gamma \rangle$$

These remarks lead to a natural way to construct the maximal BTZ-extension of  $\text{Reg}(\mathbb{E}_0^{1,2})$ , namely  $\mathbb{E}_0^{1,2}$ , by quotienting  $J^+(\Delta) = I^+(\Delta) \cup \Delta$  by  $\langle \gamma \rangle$ . The isomorphism  $\overline{\mathcal{D}}$  extends continuously to a bijective map  $\overline{\mathcal{D}} : \mathbb{E}_0^{1,2} \rightarrow J^+(\Delta)/\langle \gamma \rangle$ , by defining  $\overline{\mathcal{D}}(\tau, 0, 0) = (\tau, \tau, 0)$ . However, **if  $J^+(\Delta)$  is endowed with the usual topology, this map is not a homeomorphism**. This can be seen by taking a sequence of points tending toward  $\Delta$  following a horizontal circle intersecting  $\Delta$ . The  $\tau$  coordinate of the preimage of this sequence goes to  $-\infty$ . A thinner topology is needed on  $J^+(\Delta)$  in order to proceed.

**Definition 3.1** (BTZ-topology). *Let  $\Delta$  be a lightlike line in  $\mathbb{E}^{1,2}$ . Define the BTZ topology on  $J^+(\Delta)$  as the topology generated by the usual open subsets of  $J^+(\Delta)$  and the subsets of the form*

$$I^+(p) \cup ]p, +\infty[$$

where  $p \in \Delta$ .

**Proposition 3.2.** *Let  $\Delta$  be a lightlike line in  $\mathbb{E}^{1,2}$  and  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})$  fixing  $\Delta$  point-wise. The map  $\overline{\mathcal{D}} : \mathbb{E}_0^{1,2} \rightarrow J^+(\Delta)/\langle \phi \rangle$  is a homeomorphism.*

*Proof.* As mentionned before,  $\overline{\mathcal{D}}$  is a bijection. The topology of  $\mathbb{E}_0^{1,2}$  is generated by the diamonds  $\diamond_p^q := \text{Int}(J^+(p) \cap J^-(q))$  for  $p, q \in \mathbb{E}_0^{1,2}$  and the topology of  $J^+(\Delta)/\langle \phi \rangle$  is generated by the quotient topology of  $I^+(\Delta)/\langle \phi \rangle$  and open of the form  $\mathcal{U}_p := (I^+(p)/\langle \phi \rangle) \cup ]p, +\infty[$ .

A direct computation gives  $\mathcal{D}^{-1}(\mathcal{U}_p) = \text{Int}(J^+(\mathcal{D}^{-1}(p)))$  which is open. Then, from Corollary 16 in [Bru16],  $\mathcal{D}$  is continuous. Furthermore, for  $p \in \Delta$  and  $q \in \mathbb{E}_0^{1,2}$ ,

$$\mathcal{D}(\diamond_p^q) = \mathcal{U}_p \setminus \bigcap_{x \in \diamond_p^q} J^+(\mathcal{D}(x))$$

then  $\mathcal{D}(\diamond_p^q)$  is open. Again from Corollary 16 in [Bru16], we deduce that  $\mathcal{D}$  is open.  $\square$

Let  $M$  be Cauchy-complete globally hyperbolic spacetime and let  $\Gamma := \pi_1(M)$ . A natural construction of the maximal BTZ-extension of  $M$  would then be to consider its developpement  $\Omega$  in  $\mathbb{E}^{1,2}$  and its holonomy  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$ . For  $\gamma \in \Gamma$  such that  $\rho(\gamma)$  is parabolic, if  $\text{Fix}(\rho(\gamma)) \cap \partial\Omega \neq \emptyset$ , we add to  $\Omega$  a lightlike ray and quotient out by  $\rho$ .

**Definition 3.3** (BTZ-line associated to parabolic isometry). *Let  $G$  be a discrete torsionfree subgroup of  $\text{Isom}(\mathbb{E}^{1,2})$  and  $\Omega$  a  $G$ -invariant regular domain.*

*Let  $\phi \in G$  parabolic, define the associated BTZ-line  $\Delta_\phi$  as the interior of  $\text{Fix}(\phi) \cap \partial\Omega$  in  $\text{Fix}(\phi)$ .*

**Remark 3.4.** *The BTZ-line associated to a parabolic isometry may be empty.*

**Definition 3.5** (Augmented regular domain). *Let  $G$  be a discrete torsionfree subgroup of  $\text{Isom}(\mathbb{E}^{1,2})$  and  $\Omega$  a  $G$ -invariant regular domain. Define*

$$\widetilde{\text{Sing}}_0(\Omega, G) = \bigcup_{\phi \in G \text{ parabolic}} \Delta_\phi$$

and

$$\widetilde{\Omega}(G) = \Omega \cup \widetilde{\text{Sing}}_0(\Omega, G).$$

$\widetilde{\Omega}(G)$  is the augmented regular domain associated to  $(\Omega, G)$ , we endow it with the BTZ topology.

When there is no ambiguity, we shall simply write  $\widetilde{\text{Sing}}_0$  and  $\widetilde{\Omega}$ . Our aim is now to prove the following theorem.

**Theorem I.** *Let  $G$  be a discrete torsionfree subgroup of  $\text{Isom}(\mathbb{E}^{1,2})$  and let  $\Omega$  be a  $G$ -invariant regular domain .*

*Then  $\widetilde{\Omega}(G)/G$  is endowed with a  $\mathbb{E}_0^{1,2}$ -structure extending the  $\mathbb{E}^{1,2}$ -structure of  $\Omega/G$  and is isomorphic  $\text{BTZ} - \text{ext}(\Omega/G)$ .*

*Proof for type (i – iii) spacetimes.* If the group  $G$  only consists of spacelike translations. On the one hand  $\widetilde{\Omega} = \Omega$ . On the other hand, since the holonomy of a neighborhood of a BTZ-line is parabolic,  $\Omega/G$  is BTZ-maximal. The results follows. Case (i) is then proved as well as cases (ii) and (iii) with group generated by a spacelike translation.

Assume case (iii) with group generated by a linear parabolic isometry  $\phi$ . In this case,  $\Omega$  is  $I^+(\Pi)$  for some lightlike plane  $\Pi$  and  $\Omega/G$  is an annulus  $\{(\tau, r, \theta) \mid r \in ]R_0, +\infty[ \} \subset \mathbb{E}_0^{1,2}$ . If  $r > 0$ , then  $\Omega/G$  is BTZ-maximal and no lightlike line in  $\Pi$  is pointwise fixed by  $\phi$ . Then  $\widetilde{\Omega} = \Omega$  and the results follows. If  $r = 0$ , then there exists a lightlike line  $\Delta$  in  $\Pi$  point wise fixed by  $G$ , then  $\Omega = I^+(\Delta)$  and  $\widetilde{\Omega} = J^+(\Delta)$ . We have  $\Omega/G = \text{Reg}(\mathbb{E}_0^{1,2})$  and the results follows from Proposition 3.2.

Case (ii) with parabolic generator is treated the same way.  $\square$

### 3.1.2 Time functions on augmented regular domains

We give ourselves a discrete torsionfree isometry subgroup  $G \subset \text{Isom}(\mathbb{E}^{1,2})$  and a  $G$ -invariant regular domain  $\Omega$ . Write  $\widetilde{\Omega} = \widetilde{\Omega}(G)$ ,  $\widetilde{\text{Sing}}_0 = \widetilde{\text{Sing}}_0(\Omega, G)$  and  $\text{Sing}_0 := \widetilde{\text{Sing}}/G$ . Let  $M := \Omega/G$  and  $\overline{M} := \widetilde{\Omega}/G$ . **In this section, we assume  $M$  to be a type (iv) spacetime.**

Let  $T : \Omega \rightarrow \mathbb{R}_+^*$  the lift of the Cosmological time function of  $M$  defined in [AGH98]. As mentioned in [BB09] Theorem 1.4.1,  $T$  is a  $\mathcal{C}^1$  Cauchy time function.

**Lemma 3.6.** *The cosmological  $T$  extends continuously to  $\widetilde{\Omega}$  to the map :*

$$\tilde{T}: \begin{cases} \tilde{\Omega} & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \begin{cases} T(p) & \text{if } p \in \Omega \\ 0 & \text{if } p \in \widetilde{\text{Sing}}_0 \end{cases} \end{cases}$$

*Proof.* Let  $p \in \widetilde{\text{Sing}}_0$  and let  $\Pi_1$  the lightlike support plane of  $\Omega$  at  $p$ . Since  $\Omega/G$  is type (iv),  $\Omega$  is in the future of some spacelike plane  $\Pi_2$ . As described in [BB09], for  $q \in \Omega$ ,  $T(q)$  is length of the longest past timelike geodesic from  $q$ . As  $q$  goes to  $p$ , the past timelike geodesics from  $q$  in the domain  $J^+(\Pi_1) \cap J^+(\Pi_2) \cap J^-(q)$  tends to a segment of lightlike geodesic. Then the length of the longest past timelike geodesic goes to zero and  $\lim_{q \rightarrow p} T(q) = 0$ .  $\square$

The problem is that  $\tilde{T}$  is not a time function. Indeed,  $\tilde{T}$  is non-decreasing for the causal order on  $\tilde{\Omega}$  but not increasing. For any  $G$ -invariant measure  $\alpha$  supported on  $\widetilde{\text{Sing}}_0$  and any positive real  $a$ , we define a function

$$T_{\alpha,a}: \begin{cases} \tilde{\Omega} & \longrightarrow \mathbb{R}_+^* \cup \{+\infty\} \\ p & \longmapsto \alpha(J^-(p)) + a\tilde{T}(p) \end{cases}$$

Since  $\alpha$  is a  $G$ -invariant measure,  $T_{\alpha,a}$  descends to a function on  $M$ . Furthermore, if  $\alpha$  is absolutely continuous with respect to Lebesgue measure on  $\widetilde{\text{Sing}}_0$  and  $T_{\alpha,a} < +\infty$  then  $T_{\alpha,a}$  is increasing thus a time function. If moreover,  $\alpha(\Delta) = +\infty$  for every BTZ-line  $\Delta$  then  $T_{\alpha,a}$  will be a Cauchy-time function. One could choose the Lebesgue measure on some BTZ-line  $\Delta$  and then sums all the translation by  $G/\text{Stab}(\Delta)$ , however this gives an infinite  $T_{\alpha,a}$  in general. We thus consider such measures with cut-off below a certain point and the sum their translations. Lemmas 3.11 and 3.8 will ensure the sum induces a well defined, finite  $T_{\alpha,a}$ .

**Definition 3.7.** Let  $\Delta$  be a line of  $\mathbb{E}^{1,2}$ . Define  $G_\Delta$  the set of elements of  $G$  stabilising  $\Delta$  point-wise.

$$G_\Delta := \bigcap_{p \in \Delta} \text{Stab}_G(p)$$

**Lemma 3.8.** Let  $\Delta$  be a lightlike line such that  $G_\Delta \neq \{1\}$ . Then  $G_\Delta$  only contains parabolic isometries and an element of  $G$  stabilizing  $\Delta$  set-wise is in  $G_\Delta$ .

*Proof.* To begin with, since  $\Omega/G$  is type (iv), then, by Theorem 2.2,  $G$  and  $L(G)$  are discrete and  $L|_G$  is injective. An isometry which stabilises  $\Delta$  point wise is conjugated to a linear isometry fixing point-wise a lightlike line. It is thus conjugated to a linear parabolic isometry. Then,  $G_\Delta$  is a non trivial discrete subgroup of  $\bigcap_{p \in \Delta} \text{Stab}_{\text{Isom}(\mathbb{E}^{1,2})}(p) \simeq \mathbb{R}$ , then  $G_\Delta$  is monogene. Let  $\phi_\Delta$  be a generator of  $G_\Delta$ .

Let  $\phi \in G \setminus G_\Delta$  be such that  $\phi \cdot \Delta = \Delta$ . Let  $P \leq Q$  in  $\Delta$  be such that  $\phi P = Q$ . Up to conjugating by a translation, we can assume  $P = O$  the origin of  $\mathbb{E}^{1,2}$  so that  $\phi_L P = P$ ,  $\tau_\phi = \overrightarrow{PQ}$  and  $\phi_\Delta$  is linear. Since  $\phi \Delta = \Delta$ ,  $\phi_L \vec{\Delta} = \vec{\Delta}$  and thus  $\vec{\Delta}$  is a lightlike eigen direction of  $\phi_L$ . Then  $\phi$  is parabolic or hyperbolic. The group generated by  $\phi_L$  and  $\phi_\Delta$  is a discrete subgroup of  $\text{SO}_0(1,2)$  fixing a point of the boundary of  $\mathbb{H}^2$ , it is thus monogeneous and let  $\psi$  be a generator. There exists  $p, q \in \mathbb{Z}$  such that  $\psi^p = \phi_\Delta$  and  $\psi^q = \phi_L$ . Since  $\phi_\Delta$  is parabolic, so is  $\psi$  and thus so is  $\phi_L$ . We have  $\phi_\Delta^q \phi^p = \tau_\phi$  then  $\tau_\phi \in G$ . Since  $L \circ \rho$  is faithful and  $L(\tau_\phi) = 0$  we have  $\tau_\phi = 0$ , then  $\phi = \phi_L$  and thus  $\phi \in G_\Delta$ .  $\square$

**Corollary 3.9.** Let  $\Delta$  be a BTZ-line of  $\tilde{\Omega}$  and let  $\psi \in G$ . If there exists  $p \in \Delta$  such that  $\psi p \in \Delta$  and  $\psi p \neq p$  then  $\psi \in G_\Delta$ .

*Proof.* Let  $q \in \Delta$ ,  $\psi q = \psi(q-p+p) = \psi_L(q-p) + \psi p$ . Since  $\psi p$  and  $\psi q$  are in  $\text{Sing}_0$ , either  $\psi p - \psi q$  is spacelike or  $p, q$  belongs to the same BTZ-line. The former is not possible since  $q-p$  is lightlike, therefore  $\psi q \in \Delta$ . Then,  $\psi$  stabilises  $\Delta$  set-wise and from Lemma 3.8,  $\psi \in G_\Delta$ .  $\square$

**Lemma 3.10.** *Let  $\Delta$  be a BTZ-line of  $\widetilde{\Omega}$ . For all  $p \in \Delta$ , there exists  $\lambda > 0$  such that*

$$\forall q \in \widetilde{\Omega}, \quad \#(Gp \cap J^-(q)) \leq (1 + \lambda \widetilde{T}(q))^2$$

*Proof.* Let  $q \in \widetilde{\Omega}$ , if  $q \in \widetilde{\text{Sing}}_0$  by Lemma 3.8,  $\#(Gp \cap J^-(q)) \leq 1$ .

Let  $p \in \Delta$ , let  $p_* = \inf(\Delta)$  and let  $u = p - p_*$ . The vector  $u$  is future lightlike vector and  $\Delta = p_* + \mathbb{R}_+^* u$ . For  $v$  lightlike, define  $h_v = J^+(v) \cap \mathbb{H}^2$ . The set  $\{h_{tv} : t > 0\}$  is exactly the set of horocycles centered at  $v$ . Since  $L(G)$  is discrete,  $\mathbb{H}^2/L(G)$  is a complete  $\mathbb{H}^2$ -manifold and there exists an embedded horocycle around the cusp associated to  $u$ . Let  $\lambda > 0$  such that  $h_{\lambda u}$  is embedded. Let  $\vec{n}$  be the vertical future unit timelike vector. Let  $\phi \in G$

$$\phi p \in J^-(q) \Leftrightarrow \phi p_* + \phi_L u \in J^-(q) \quad (1)$$

$$\Leftrightarrow \phi_L u \in J^-(q - \phi p_*) \quad (2)$$

$$\Rightarrow |\langle \phi_L u | \vec{n} \rangle| \leq |\langle q - \phi p_* | \vec{n} \rangle| \quad (3)$$

$$\Rightarrow |\langle \phi_L u | \vec{n} \rangle| \leq T(q) \quad (4)$$

On the one hand, for  $v$  lightlike, the stereographic projection of  $\mathbb{H}^2$  onto the Poincaré disc on  $\vec{n}^\perp$  sends an horocycle  $J^+(v) \cap \mathbb{H}^2$  to a Euclidean circle of radius  $(1 + |\langle v | \vec{n} \rangle|)^{-1}$ . On the other hand, the horocycles  $h_{\lambda \phi_L u}$  are disjoint for  $\phi_L \in L(G)/L(G_\Delta)$ . If  $\phi p \in J^-(q)$ , then the radius of  $h_{\lambda \phi_L u}$  is greater than  $(1 + \lambda T(q))^{-1}$ . Since the total area of the disjoint horoball is less than  $\pi$ , there exists at most  $(1 + \lambda T(q))^2$  such  $\phi_L \in L(G)/L(G_\Delta)$ . Since  $\Omega/G$  is type  $(iv)$ , then  $L|_G$  is injective and the result follows.  $\square$

**Corollary 3.11.** *Let  $\phi \in G$  parabolic. Then for all  $p \in \Delta_\phi$ ,  $Gp$  is discrete.*

**Proposition 3.12.** *There exists a measure  $\alpha \widetilde{\text{Sing}}_0$  such that for all  $a \in \mathbb{R}_+^*$ ,*

- $T_{\alpha,a}$  is  $\mathcal{C}^1$  on  $\Omega$  and  $\mathcal{C}^0$  on  $\widetilde{\Omega}$ ;
- $T_{\alpha,a}$  is  $G$ -invariant.
- $T_{\alpha,a}$  is a Cauchy time function on  $\widetilde{\Omega}$

*Proof.* Choose a set of representative  $(\Delta_i)_{i \in I}$  of  $\widetilde{\text{Sing}}_0$ . The set  $I$  is countable we can thus assume  $I \subset \mathbb{N}$  and for each  $i \in I$ , choose a decreasing sequence  $(p_n^{(i)})_{n \in \mathbb{N}} \in \Delta_i^\mathbb{N}$  such that  $\lim_{n \rightarrow +\infty} p_n^{(i)} = \min(\Delta_i)$ . Let  $N(i, n)$  be the number of triplet  $(j, k, \psi)$  with  $j \leq i$  and  $k \leq n$  and  $\psi \in G/G_{\Delta_j}$  such that  $\psi p_k^j \in \Delta_i$ . We can choose a family  $(\varphi_n^{(i)})_{n \in \mathbb{N}, i \in I}$  such that for all  $n \in \mathbb{N}$  and  $i \in I$ ,

$$(i) \quad \varphi_n^{(i)} \text{ is in } \mathcal{C}^1(\Delta_i, \mathbb{R}_+),$$

$$(ii) \quad \|\varphi_n^{(i)}\|_{\mathcal{C}^1} \leq 1$$

$$(iii) \quad \lim_{x \rightarrow +\infty} \varphi_n^{(i)}(x) = 1$$

$$(iv) \quad \forall x \in \Delta_i, \varphi_n^{(i)}(x) = 0 \Leftrightarrow x \leq p_n^{(i)}$$

Choose a geodesic parametrisation of  $\Delta_i$  for each  $i \in I$  and let  $\lambda_i$  the image of the Lebesgue measure on  $\mathbb{R}_+^*$  by this parametrisation. From Lemma 3.10, for  $n \in \mathbb{N}$  and  $i \in I$ , let  $\mu_n^{(i)} \geq 1$  be such that

$$\forall q \in \Omega, \quad \# \{ \phi \in G/G_\Delta \mid \phi p_n^{(i)} \in J^-(q) \} \leq (1 + \mu_n^{(i)} T(q))^2$$

Let

$$\alpha = \sum_{i \in I} \sum_{\psi \in G/G_{\Delta_i}} \sum_{n \in \mathbb{N}} \omega_n^{(i)} \psi \# (\varphi_n^{(i)} \lambda_i)$$

where  $\omega_n^{(i)} = \frac{2^{-i-n}}{\lambda_i(J^-(p_0^{(i)})) \mu_n^{(i)}}$ .

Define for  $i \in I$  and  $n \in \mathbb{N}$ ,

$$\alpha_n^{(i)} : \begin{cases} \tilde{\Omega} & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \sum_{\psi \in G/G_\Delta} \varphi_n^{(i)} \lambda_i(J^-(\psi p)) \end{cases}$$

The sum is locally finite thus  $\alpha_n^{(i)}$  is  $\mathcal{C}^1$  and finite. Furthermore, for all  $q \in \tilde{\Omega}$  :

$$\|\alpha_n^{(i)}\|_{\mathcal{C}^1(J^-(q))} \leq \lambda_i(J^-(p_0^{(i)})) (1 + \mu_n^{(i)} T(q))^2.$$

Then, for all  $q \in \tilde{\Omega}$  :

$$\sum_{i \in I} \sum_{n \in \mathbb{N}} \|\omega_n^{(i)} \alpha_n^{(i)}\|_{\mathcal{C}^1(J^-(q))} \leq \sum_{i \in I} \sum_{n \in \mathbb{N}} 2^{-i-n} (1 + T(q))^2 = 4(1 + T(q))^2.$$

Thus, the sum  $\sum_{i \in I} \sum_{n \in \mathbb{N}} \omega_n^{(i)} \alpha_n^{(i)}$  is normally convergent on compact subset of  $\tilde{\Omega}$  for the  $\mathcal{C}^1$  norm and is thus  $\mathcal{C}^1$ .

It remains to prove  $T_{\alpha,a}$  is Cauchy, i.e that  $T_{\alpha,a}$  is surjective and increasing on inextendible causal curves. Let  $c : \mathbb{R} \rightarrow \tilde{\Omega}$  be an inextendible causal curve, define  $\Delta = c \cap \widetilde{\text{Sing}}_0$  and  $c^0 = c \cap \Omega$ . The two pieces  $\Delta$  and  $c^0$  are connected and  $\Delta$  is in the past of  $c^0$ . The function  $T$  is increasing on  $c^0$  then so is  $T_{\alpha,a}$ . Since  $\alpha$  is absolutely continuous with respect to Lebesgue measure on  $\widetilde{\text{Sing}}_0$  then  $T_{\alpha,a}$  is increasing on  $\Delta$ . When  $t \rightarrow -\infty$ ,  $\tilde{T}(c(t))$  and  $\alpha(J^-(c(t)))$  go to 0 thus  $T_{\alpha,a}(c(t))$  goes to 0. If  $c^0 = \emptyset$ , then  $\bigcup_{t>0} J^-(c(t))$  is a connected component component of  $\widetilde{\text{Sing}}_0$  and by condition (iii),  $\alpha(J^-(c(t)))$  goes to  $+\infty$ . If  $c^0 \neq \emptyset$ , then for  $c^0$  is a non-empty inextendible future causal curve of  $\Omega$  and thus  $\lim_{t \rightarrow +\infty} T(c(t)) = +\infty$ . In any case,  $\lim_{t \rightarrow +\infty} T_{\alpha,a}(c(t)) = +\infty$ . Finally,  $T_{\alpha,a}$  is a Cauchy time function on  $\tilde{\Omega}$ .  $\square$

### 3.1.3 Construction of the maximal BTZ-extension of a regular domain

We give ourselves a discrete torsionfree isometry subgroup  $G \subset \text{Isom}(\mathbb{E}^{1,2})$  and a  $G$ -invariant regular domain  $\Omega$ . We assume  $\Omega/G$  is of type (iv). Write (see Definition 3.5)  $\tilde{\Omega} = \tilde{\Omega}(G)$ ,  $\widetilde{\text{Sing}}_0 = \widetilde{\text{Sing}}_0(\Omega, G)$ . Let  $M := \Omega/G$ ,  $\overline{M} := \tilde{\Omega}/G$  with the quotient topology and let  $\pi : \tilde{\Omega} \rightarrow \overline{M}$  be the natural projection.

**Proposition 3.13.**  $\overline{M}$  is Hausdorff.

*Proof.* Let  $\alpha$  be a measure given by Proposition 3.12 and let  $a \in \mathbb{R}_+^*$ . Let  $p, q \in \tilde{\Omega}$  be such that  $\pi(p) \neq \pi(q)$ .

- If  $p$  and  $q$  are in  $\Omega$ , then  $\pi(p)$  and  $\pi(q)$  are in  $M$  and since  $M$  is Hausdorff, then  $\pi(p)$  and  $\pi(q)$  are separated in  $M$ . And thus in  $\pi(p)$  and  $\pi(q)$  are separated in  $\overline{M}$ .

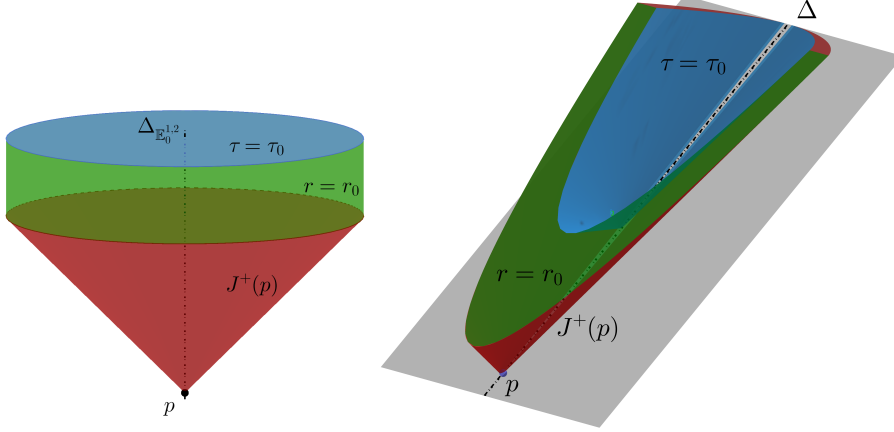


Figure 1: Tubular neighborhood of a BTZ point and its development  
On the left, a tubular subset of  $\mathbb{E}_0^{1,2}$ . On the right its development into  $\mathbb{E}^{1,2}$ . Colors are associated to remarkable sub-surfaces and their developments.

- If  $p$  and  $q$  are in  $\widetilde{\text{Sing}}_0$ , then either  $T_{\alpha,a}$  separates  $p$  and  $q$  or  $p, q$  are not on the same BTZ-line. In the latter case we can multiply  $\alpha$  by a  $G$ -invariant function equal to  $1/2$  on the orbit of the BTZ-line of  $p$  and  $1$  on the other BTZ-lines to obtain  $T_{\alpha',a}(p) \neq T_{\alpha',a}(q)$ . Since  $T_{\alpha,a}$  is  $G$ -invariant and continuous,  $\pi(p)$  and  $\pi(q)$  are separated.
- If  $p \in \Omega$  and  $q \in \widetilde{\text{Sing}}$ , then one can change the  $a$  parameter to obtain  $T_{\alpha,a}(p) \neq T_{\alpha,a}(q)$ . Then we can use the same argument as before.

□

**Proposition 3.14.**  $\overline{M}$  has a  $\mathbb{E}_0^{1,2}$ -structure such that  $\text{Reg}(\overline{M}) = M$ . Furthermore, this structure is globally hyperbolic.

*Proof.* The  $\mathbb{E}^{1,2}$ -atlas of  $M$  gives an atlas of  $\overline{M} \setminus (\widetilde{\text{Sing}}_0)/G$ , it suffices to construct charts around points of  $\widetilde{\text{Sing}}_0/G$ . Beware that hereafter, the topology on  $\widetilde{\Omega}$  is the BTZ-topology given in Definition 3.1. Let  $\Delta$  be a connected component of  $\widetilde{\text{Sing}}_0$  (i.e. a BTZ-line of  $\widetilde{\Omega}$ ) and  $p \in \Delta$  be some point on it. Let  $G^* := G \setminus G_\Delta$ . We can choose the map  $\overline{D}$  of Proposition 3.2 such that  $\Delta \subset \overline{D}(\text{Sing}_0(\mathbb{E}_0^{1,2}))$ . Denote  $q = \overline{D}^{-1}(p)$ . We now construct a  $G_\Delta$ -invariant neighborhood  $\mathcal{W}$  of  $p$ , open for the BTZ-topology and disjoint from its image by  $G^*$ . Then  $\overline{D}$  will induce a homeomorphism between some open neighborhood of  $q = \overline{D}^{-1}(p)$  and  $\mathcal{W}/G_\Delta$ .

Let  $q_* \in J^-(q)$  and denote  $p_* = \overline{D}(q)$  so that  $p_* \in J^-(p)$  and let

$$\mathcal{V} = \text{Int}(J^+(q_*) \cap \{r < R, \tau < \tau^*\}) \quad \text{and} \quad \mathcal{U} = \pi^{-1}(\overline{D}(\mathcal{V}))$$

where  $(\tau, r, \theta)$  are the cylindrical coordinates of  $\mathbb{E}_0^{1,2}$  and  $R > 0$  and  $\tau^* > \frac{1}{2}R$ . See Figure 3.1.3 below which depicts  $\mathcal{V}$  and  $\mathcal{U}$ . Now, choose  $\mathcal{U}_0 \subset \mathcal{U}$  some relatively compact open domain such that  $G_\Delta \cdot \mathcal{U}_0 = \mathcal{U}$ , then:

$$\mathcal{U} \setminus G^*\mathcal{U} = \mathcal{U} \setminus (G^*\mathcal{U} \cap \mathcal{U}) \tag{5}$$

$$= (G_\Delta \mathcal{U}_0) \setminus G_\Delta((G^*\mathcal{U}) \cap \mathcal{U}_0) \tag{6}$$

$$= G_\Delta(\mathcal{U}_0 \setminus (G^*\mathcal{U}) \cap \mathcal{U}_0) \tag{7}$$



Since  $\mathcal{U} \subset J^+(p_*)$ , for  $\phi \in G$ , if  $(\phi\mathcal{U}) \cap \mathcal{U}_0 \neq \emptyset$  then  $\phi p_* \in J^-(\mathcal{U}_0) \subset J^-(\overline{\mathcal{U}_0})$ . From Corollary 3.11 and since  $J^-(\overline{\mathcal{U}_0})$  is compact, the set  $Gp_* \cap J^-(\overline{\mathcal{U}_0})$  is finite, say  $\{\phi_0 p_*, \phi_1 p_*, \dots, \phi_n p_*\}$  with  $\phi_0 = 1$ . Therefore :

$$\mathcal{U} \setminus G^*\mathcal{U} = G_\Delta \cdot (\mathcal{U}_0 \setminus (\mathcal{U}_0 \cap G^*\mathcal{U})) \quad (8)$$

$$= G_\Delta \cdot \left( \mathcal{U}_0 \setminus \bigcup_{i=1}^n \phi_i \mathcal{U} \right) \quad (9)$$

$$\supset G_\Delta \cdot \left( \mathcal{U}_0 \setminus \bigcup_{i=1}^n J^+(\phi_i p_*) \right) \quad (10)$$

Notice that from Lemma 3.8, for all  $\phi \in G$  such that  $\phi p_* \in J^-(p_*)$ , we have  $\phi \in G_\Delta$ . Thus the only  $\phi_i$  such that  $\phi_i p_* \in J^-(\overline{\mathcal{U}_0})$  is  $\phi_0 = 1$ . Let  $\mathcal{W} := G_\Delta (\mathcal{U}_0 \setminus \bigcup_{i=1}^n J^+(\phi_i p_*))$ , then  $\mathcal{W}$  is an open subset of  $\mathcal{U}$ , disjoint from  $G^*\mathcal{W}$  and containing  $p$ .

A function  $T_{\alpha,a}$  from Proposition 3.12 is a  $G$ -invariant Cauchy-function of  $\tilde{\Omega}$ . It thus induces a Cauchy function of  $\bar{M}$  which proves that  $\bar{M}$  is globally hyperbolic.  $\square$

**Theorem I.** *Let  $G$  be a discrete torsionfree subgroup of  $\text{Isom}(\mathbb{E}^{1,2})$  and let  $\Omega$  be a  $G$ -invariant regular domain .*

*Then  $\tilde{\Omega}(G)/G$  is endowed with a  $\mathbb{E}_0^{1,2}$ -structure extending the  $\mathbb{E}^{1,2}$ -structure of  $\Omega/G$  and is isomorphic  $\text{BTZ} - \text{ext}(\Omega/G)$ .*

*Proof for type (iv) spacetimes.* Propositions 3.13 and 3.14 prove that  $\Omega/G \rightarrow \tilde{\Omega}/G$  is a BTZ-embedding. It remains to prove that  $\tilde{\Omega}/G$  is BTZ-maximal. Consider the maximal BTZ-embedding  $\tilde{\Omega}/G \xrightarrow{i} N$ , take a point  $p \in \text{Sing}_0(N)$ , a compact diamond neighborhood  $\mathcal{U}$  around  $p$  included in a chart around  $p$  and some loop  $c := \{r = R_0, \tau = \tau_0\}$  around the line  $\text{Sing}_0(\mathcal{U})$ . Let  $\mathcal{D}$  be the developping map of  $\text{Reg}(N)$ . The image of the holonomy of  $\text{Reg}(\mathcal{U})$  is generated by a parabolic isometry  $\phi$  fixing a lightlike line  $\Delta$  which intersects the boundary of  $\mathcal{D}(\text{Reg}(\mathcal{U}))$  along a segment  $[p_*, p^*]$ . Let  $\mathcal{V} := \mathcal{D}(\text{Reg}(\mathcal{U}))$ , we can assume that  $(G \setminus G_\Delta)\mathcal{V} \cap \mathcal{V} = \emptyset$ . By Proposition 3.2,  $\mathcal{D}$  induces a homeomorphism  $\bar{\mathcal{D}} : \mathcal{U} \rightarrow (\mathcal{V} \cup [p_*, p^*])/G_\Delta$  and  $i|_{\mathcal{V}/G_\Delta}$  is a continuous section of  $\bar{\mathcal{D}}$  on  $\mathcal{V}/G_\Delta$ . Therefore, by continuity,  $\bar{\mathcal{D}}^{-1} = i|_{(\mathcal{V} \cup [p_*, p^*])/G_\Delta}$  and  $p$  is in the image of  $i$ . Finally,  $i$  is surjective.  $\square$

### 3.1.4 Absolutely maximal singular spacetimes

We introduce the notion of absolutely maximal  $\mathbb{E}_A^{1,2}$ -manifolds which will prove relevant in our description of Lorentzian Moduli spaces. See Definition 21 in [Bru16] for a definition of  $\mathbb{E}_A^{1,2}$ -manifolds.

**Definition 3.15.** *A connected Cauchy-complete globally hyperbolic  $\mathbb{E}_A^{1,2}$ -manifold  $M_1$  is absolutely-maximal if all  $\mathbb{E}_A^{1,2}$ -embedding  $M_1 \rightarrow M_2$ , with  $M_2$  Cauchy-complete globally hyperbolic and connected, is onto.*

**Remark 3.16.** *Beware that the absolute maximality depends strongly on the category of  $\mathbb{E}_A^{1,2}$ -manifold you consider. Indeed, an absolutely maximal  $\mathbb{E}^{1,2}$ -manifold may have a BTZ-extension and thus not absolutely as  $\mathbb{E}_0^{1,2}$ -manifold. For instance :  $\text{Reg}(\mathbb{E}_0^{1,2})$  is absolutely maximal as  $\mathbb{E}^{1,2}$ -manifold but embeds into  $\mathbb{E}_0^{1,2}$ , thus  $\text{Reg}(\mathbb{E}_0^{1,2})$  is not absolutely maximal as  $\mathbb{E}_0^{1,2}$ -manifold.*

We do not claim that there exists a unique absolutely maximal extension for every  $\mathbb{E}_A^{1,2}$ -manifold. A theorem of existence and unicity is known for Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold as a consequence of Theorem 2.2. We now use Theorem I to extend it for Cauchy-complete  $\mathbb{E}_0^{1,2}$ -manifolds.

**Proposition 3.17.** *Let  $M$  be a connected Cauchy-complete globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifold then there exists a  $\mathbb{E}_0^{1,2}$ -manifold  $\overline{M}$  absolutely maximal in which  $M$  embeds. Moreover,  $\overline{M}$  is unique up to isomorphism.*

*Proof.* Let  $M_1$  be the maximal Cauchy-extension of  $M$ , it exists and is unique (see for instance [BBS11]). From Theorem 2.4,  $\text{Reg}(M_1)$  is a Cauchy-complete and Cauchy-maximal globally hyperbolic  $\mathbb{E}^{1,2}$ -manifold. Let  $M_2$  be the maximal BTZ-extension of the absolutely maximal extension of  $\text{Reg}(M_1)$  among  $\mathbb{E}^{1,2}$ -manifolds. Let  $\tilde{\Omega}_1$  (resp.  $\tilde{\Omega}_2$ ) be the augmented regular domain associated to  $\text{Reg}(M_1)$  (resp.  $\text{Reg}(M_2)$ ). These augmented regular domain can be chosen such that  $\tilde{\Omega}_1 \subset \tilde{\Omega}_2$ . From Theorem I, this inclusion induces an embedding  $i : M_1 \rightarrow M_2$ . Let  $M_3$  be a connected Cauchy-complete Cauchy-maximal  $\mathbb{E}_0^{1,2}$ -manifold and an embedding  $j : M \rightarrow M_3$ . We can extend  $j$  to an embedding  $M_1 \rightarrow M_3$ . We thus obtain an embedding  $\text{Reg}(M_1) \rightarrow \text{Reg}(M_3)$  and thus by absolute maximality of  $\text{Reg}(M_2)$ , an embedding  $\text{Reg}(M_3) \rightarrow \text{Reg}(M_2)$ . Then we obtain an inclusion  $\tilde{\Omega}_3 \subset \tilde{\Omega}_2$  with  $\tilde{\Omega}_3$  the augmented regular domain associated to  $\text{Reg}(M_3)$ . By Theorem I, we then get an embedding  $M_3 \rightarrow M_2$  such that the following diagram commutes

$$\begin{array}{ccc} M & \longrightarrow & M_1 \\ \downarrow j & & \downarrow i \\ M_3 & \longrightarrow & M_2 \end{array}$$

If we have an embedding  $f : M_2 \rightarrow M_3$  then  $M$  embeds into  $M_3$  and thus we obtain a map  $g : M_3 \rightarrow M_2$ . The commutative diagram then implies that  $g \circ f = \text{Id}_{M_2}$  and thus  $f$  is surjective. Thus  $M_2$  is absolutely maximal. If  $M_3$  is absolutely maximal then the map  $M_3 \rightarrow M_2$  is surjective thus an isomorphism. Thus  $M_2$  is unique up to isomorphism.  $\square$

**Lemma 3.18.** *Let  $M$  be a  $\mathbb{E}_A^{1,2}$  manifold, there exists a vector field on  $M$  such that*

- $X$  is  $\mathcal{C}^1$  and non singular;
- $X$  is future causal;
- For  $p \in \text{Sing}(M)$ ,  $X_p$  is parallel to the direction of the singular line through  $p$ .

*Sketch of proof.* Let  $\mathcal{V} = \text{Reg}(M)$ ,  $\mathcal{V}$  has a time order relation. Thus, from Lemma 32 p145 of [O’N83], there exists a timelike hence causal and non-singular  $\mathcal{C}^1$  vector field  $X_{\mathcal{V}}$  on  $\mathcal{V}$ . Take a family of disjoint chart neighborhoods  $(\mathcal{U}_i)_{i \in I}$  of the singular lines  $(\Delta_i)_{i \in I}$ . On each  $\mathcal{U}_i$ , define  $X_i$  a constant vector field parallel to  $\Delta_i$  then construct a partition of unity  $(\varphi_{\mathcal{U}_i} : i \in I, \varphi_{\mathcal{V}})$  associated to the open cover  $(\mathcal{U}_i : i \in I, \mathcal{V})$  such that  $\varphi_{\mathcal{U}_i} = 1$  on a neighborhood of  $\Delta_i$ . The vector field  $\sum_{i \in I} \varphi_{\mathcal{U}_i} X_i + \varphi_{\mathcal{V}} X_{\mathcal{V}}$  satisfies the wanted properties.  $\square$

**Lemma 3.19.** *Let  $M$  be a globally hyperbolic  $\mathbb{E}_A^{1,2}$ -manifold. If  $M$  is Cauchy-compact, then the following are equivalent*

- (i)  $M$  is Cauchy-maximal;
- (ii)  $M$  is absolutely maximal.

*Proof.* Assume  $M$  is Cauchy-compact.

- Assume  $M$  is absolutely maximal. Since a Cauchy-embedding is an embedding of  $\mathbb{E}_A^{1,2}$ -manifold, in particular every Cauchy-embedding is onto and thus  $M$  is Cauchy-maximal.

- Assume  $M$  is Cauchy-maximal and Consider  $M \xrightarrow{i} M'$  a  $\mathbb{E}_A^{1,2}$ -embedding with  $M'$  globally hyperbolic. Take  $\Sigma_1$  (resp.  $\Sigma_2$ ) a spacelike Cauchy-surface of  $M_1$  (resp.  $M_2$ ), such a surface exists from Theorem 1 in [Bru16]. Take  $X$  a vector field on  $M'$  given by Lemma 3.18. For every  $p \in \Sigma_1$ , the line of flow of  $X$  through  $p$  intersects  $\Sigma_2$  exactly once. The map  $f : \Sigma_1 \rightarrow \Sigma_2$  defined this way is a local homeomorphism. Since  $\Sigma_1$  is compact, then  $f$  is proper,  $f$  is thus a covering and thus  $f^* : \pi_1(\Sigma_1) \rightarrow \pi_1(\Sigma_2)$  is surjective. Finally,  $i^* : \pi_1(M_1) \rightarrow \pi_1(M_2)$  is onto. The proof of Lemma 45 p427 of [O'N83] applies to the context of  $\mathbb{E}_A^{1,2}$ -manifolds and from this Lemma we deduce that  $\Sigma_1$  is achronal, hence acausal.

From Lemma 43 p426 of [O'N83], the Cauchy developpement of  $\Sigma_1$  is open and, since  $\Sigma_1$  is compact, it is also closed. By connectedness of  $M_2$ , the Cauchy developpement of  $\Sigma_1$  is the whole  $M_2$  thus  $i$  is a Cauchy-embedding. However,  $M$  is Cauchy-maximal thus  $i$  is surjective.

□

**Remark 3.20.** *Neither the Cauchy-compactity nor the Cauchy-maximality of a spacetime  $M$  depend on the category of  $\mathbb{E}_A^{1,2}$ -manifold in which we are considering  $M$ . Therefore, a Cauchy-compact and Cauchy-maximal  $\mathbb{E}_A^{1,2}$ -manifold is absolutely maximal whatever the category in which it is considered.*

**Lemma 3.21.** *Let  $N$  be a Cauchy-complete Cauchy-maximal globally-hyperbolic  $\mathbb{E}^{1,2}$ -manifold and let  $\bar{N}$  be its absolutely maximal extension among  $\mathbb{E}_0^{1,2}$ -manifold.*

*If  $\bar{N}$  is Cauchy-compact, then the following are equivalent:*

- (i)  $N$  is absolutely maximal among  $\mathbb{E}^{1,2}$ -manifolds;
- (ii)  $\text{BTZ} - \text{ext}(N) = \bar{N}$
- (iii)  $\text{BTZ} - \text{ext}(N)$  contains a point of each BTZ-line of  $\bar{N}$ .

*Proof.* Let  $i : N \rightarrow \bar{N}$  be an embedding,  $i(N) \subset \text{Reg}(\bar{N})$ .

- If  $N$  is absolutely maximal, then  $i(N) = \text{Reg}(\bar{N})$  and  $\text{BTZ} - \text{ext}(N) \simeq \bar{N}$ . Thus (i)  $\Rightarrow$  (ii).
- Assume  $\text{BTZ} - \text{ext}(N) \simeq \bar{N}$ , we have  $\text{Reg}(\bar{N}) = i(N)$ . Let  $j : N \rightarrow N'$  be the absolutely maximal extension of  $N$  among  $\mathbb{E}^{1,2}$ -manifold. From Proposition 3.17, the absolutely maximal extension of  $N'$  is  $\bar{N}$  and we have an embedding  $k : N' \rightarrow \bar{N}$ . Since,  $k(N') \subset \text{Reg}(\bar{N}) = i(N)$ , then  $\phi := i^{-1} \circ k \circ j$  is an automorphism of  $N$ . Finally,  $j = k^{-1} \circ i \circ \phi$  is surjective and thud  $N = \text{Reg}(\bar{N})$  is absolutely maximal. This proves (i)  $\Leftarrow$  (ii).
- Assume  $\text{BTZ} - \text{ext}(N) = \bar{N}$ , then trivially  $\text{BTZ} - \text{ext}(N)$  contains a point of each BTZ-line of  $\bar{N}$ . Therefore, (ii)  $\Rightarrow$  (iii).
- Assume  $\text{BTZ} - \text{ext}(N)$  contains a point of each BTZ-line of  $\bar{N}$ . From Proposition 2.2,  $N$  is homeomorphic to  $\text{Reg}(\bar{N})$  and each ends of a Cauchy-surface of the latter corresponds to a BTZ-line of  $\bar{N}$ . Since  $\text{BTZ} - \text{ext}(N)$  contains a point of each BTZ-line of  $\bar{N}$ , a Cauchy-surface of  $\text{BTZ} - \text{ext}(N)$  has no ends and is thus compact. From Theorem 2.4,  $\text{BTZ} - \text{ext}(N)$  is Cauchy-maximal. Thus, from Proposition 3.19,  $\text{BTZ} - \text{ext}(N)$  is absolutely maximal and (iii)  $\Rightarrow$  (ii).

### 3.1.5 Maximal BTZ-extension of absolutely maximal spacetimes

Given a regular domain  $\Omega$  invariant under the action of some discrete torsionfree subgroup of isometries  $G \subset \text{Isom}(\mathbb{E}^{1,2})$ . From the construction above, the only remaining question is whether the line of fixed points of a parabolic element of  $G$  is in the boundary of  $\Omega$ .

**Proposition 3.22.** *Let  $\Omega$  be a regular domain invariant under the action a discrete torsionfree subgroup  $G \subset \text{Isom}(\mathbb{E}^{1,2})$  such that  $\Omega/G$  is absolutely maximal. Then, for all  $\psi \in G$  parabolic, the BTZ-line associated to  $\psi$  is non empty.*

*Proof of Proposition 3.22 for type (i – iii) .* Among the three first cases of Theorem 2.2, case (i) does not admit parabolic isometry and is thus trivial. In cases (ii) and (iii), the group  $G$  is either generated by a translation in which case the proposition is trivial or by a parabolic isometry  $\psi$ . Since  $\Omega/G$  is absolutely maximal,  $\Omega = I^+(\text{Fix}(\psi))$  and  $\Delta_\psi \neq \emptyset$ .  $\square$

Only the case (iv) of Theorem 2.2 remains in which case  $L|_G$  is injective and  $L(G)$  is discrete. **We now assume that  $\Omega/G$  is a type (iv) spacetime.**

Let  $\psi \in G$ , let  $\Delta = \text{Fix}(\psi)$  and let  $G_\Delta := \text{Stab}(\Delta)$ . We can assume  $\Delta$  goes through the origin  $O$  of  $\mathbb{E}^{1,2}$  and take some  $u$  on  $\Delta$  above and distinct from  $O$ . For  $R \in \mathbb{R}$ , defines the planes

$$\Pi_R := \{x \in \mathbb{E}^{1,2} \mid \langle x|u \rangle = -R\}$$

so that they are the the planes parallel to  $\Delta^\perp$  and we have

$$\forall R \in \mathbb{R}, I^-(\Pi_R) = \{x \in \mathbb{E}^{1,2} \mid \langle x|u \rangle > -R\}.$$

Since  $G$  acts on  $\Omega$  torsionfree,  $\Delta \cap \Omega = \emptyset$ , furthermore  $\forall x \in \Omega, J^+(x) \subset \Omega$  thus  $\Omega \subset I^+(\Delta) = I^+(\Pi_0)$ . Let

$$R_0 := \max\{R \in \mathbb{R} \mid \Pi_R \cap \Omega = \emptyset\} \quad \text{and} \quad \mathcal{U}_{\lambda,R} := I^+(\lambda u) \cap I^-(\Pi_R)$$

Let  $\mathcal{C}$  be the cone of future lightlike vector from  $O$ . Up to some translation of the origin  $O$  along  $\Delta$ , we can assume that  $\mathcal{C} \cap \Pi_{R_0+1} \subset \Omega$ .

We want to find some  $\lambda > 1$  and  $R > R_0$  such that  $\mathcal{U}_{\lambda,R}$  is disjoint from its translations by  $G \setminus G_\Delta$ .

**Lemma 3.23.** *Let  $\lambda > 1$  and  $R = R_0 + 1$ . Let  $\phi \in G$  such that  $\phi\mathcal{U}_{\lambda,R} \cap \mathcal{U}_{\lambda,R} \neq \emptyset$ , then*

$$0 \geq \langle \phi_L u | u \rangle \geq -\frac{R_0 + 2}{\lambda}$$

*Proof.* To begin with,  $\langle \phi_L u | u \rangle$  is non-positive since  $\phi_L u$  and  $u$  are both future pointing.

If  $\phi \in G_\Delta$ , then  $\phi_L u = \phi u = u$  and thus  $\langle \phi_L u | u \rangle = 0$ . If  $\phi \notin G_\Delta$ , we have  $\phi(\lambda u) \in I^-(\Pi_R)$  thus

$$\lambda \langle \phi_L u | u \rangle + \langle \tau_\phi | u \rangle > -R$$

Furthermore, there exists a unique  $v \in \mathcal{C}$  such that  $\phi_L v = \alpha v$  with  $0 < \alpha \leq 1$  and  $v \in u$ . Since  $L(G)$  is discrete, the vector  $v$  is parallel to  $u$  if and only if  $\phi \in G_\Delta$ . Then, for  $\phi \notin G_\Delta$ ,  $v \in \Pi_R \cap \mathcal{C} \subset \Omega$  and since  $G$  stabilises  $\Omega$ ,  $\phi v \in \Omega$  and  $\langle \phi v | u \rangle \leq -R_0$ . Therefore :

$$-R_0 \geq \langle \phi v | u \rangle = \alpha \langle v | u \rangle + \langle \tau_\phi | u \rangle = -R + \langle \tau_\phi | u \rangle$$

and then :

$$1 = R - R_0 \geq \langle \tau_\phi | u \rangle.$$

We thus have :

$$\langle \phi_L u | u \rangle \geq \frac{-R - 1}{\lambda} = \frac{-R_0 - 2}{\lambda}$$

$\square$

**Lemma 3.24.** *Let  $\lambda > 2$ ,  $R = R_0 + 1$  and let  $\phi \in G$ .*

*Then, there exists  $\phi_0 = 1, \phi_1, \dots, \phi_n \in G$  such that*

$$\phi \mathcal{U}_{\lambda, R} \cap \mathcal{U}_{\lambda, R} \neq \emptyset \Rightarrow \phi \in \bigcup_{i=0}^n G_{\Delta} \phi_i G_{\Delta}$$

*Proof.* Take some  $p^* \in \mathbb{E}^{1,2}$  such that  $J^+(O) \cap \Pi_R \cap J^-(p^*)$  contains a fundamental domain of the action of  $G_{\Delta}$  on  $\mathcal{C} \cap \Pi_R$ . Then,  $J^+(O) \cap J^-(p^*)$  contains a fundamental domain of the action of  $G_{\Delta}$  on  $\mathcal{C} \cap \Pi_{R'}$  for every  $R' \in ]0, R[$ . From Lemma 3.23, for all  $k \in \mathbb{Z}$ ,

$$0 \geq \langle L(\psi^k \phi)u | u \rangle > -\frac{R_0 + 2}{\lambda}$$

and  $k \in \mathbb{Z}$  can be chosen such that  $L(\psi^k \phi)u$  is in the fundamental domain of the action of  $G_{\Delta}$  on  $\mathcal{C}$  and thus in  $J^+(O) \cap J^-(p^*)$ . Thus, there exists  $k \in \mathbb{Z}$  such that  $L(\psi^k \phi).p_* \in J^-(p^*) \cap J^+(O)$ . The diamond  $J^+(O) \cap J^-(p^*)$  is compact and from [EP88]  $L(G)u$  is discrete, furthermore  $L|_G$  is injective, thus there exists only finitely many  $[\phi'] \in G/G_{\Delta}$  such that  $L(\phi')u \in J^-(p^*) \cap J^+(O)$ . Let  $\{\phi_0 = 1, \phi_1, \dots, \phi_n\}$  be a set of representative of these  $[\phi']$ , then :

$$\phi_L \in \bigcup_{i=1}^n G_{\Delta} L(\phi_i) G_{\Delta}$$

which yields the results since  $L|_G$  is injective and  $L(G_{\Delta}) = G_{\Delta}$ . □

*Proof of Proposition 3.22, type (iv).* Take

$$R = R_0 + 1 \quad \text{and} \quad \lambda = \max_{i \in [1, n]} \left( \frac{R_0 + 1}{-\langle L(\phi_i)u | u \rangle} \right) + 1$$

we obtain  $\phi \mathcal{U}_{\lambda, R} \cap \mathcal{U}_{\lambda, R} = \emptyset$  for  $\phi \in G \setminus G_{\Delta}$ .

Consider the domain

$$\Omega' := \Omega \cup \bigcup_{[\phi] \in G/G_{\Delta}} \phi \mathcal{U}_{\lambda, R}.$$

$\Omega'$  is globally hyperbolic and  $G$ -invariant. Let  $\tilde{\Sigma}$  be  $G$ -invariant Cauchy-surface of  $\Omega$ . Take  $p \in I^-(\Pi_R) \cap \Omega$ ,  $I^+(p) \cap \Pi_R \subset \Omega$  is the interior of a parabola inside  $\Pi_R$  thus every lightlike line in  $\Pi_R$  intersects  $\Omega$ . The surface  $\tilde{\Sigma}$  is a Cauchy-surface of  $\Omega$  and thus intersects every lightlike line in  $\Pi_R$ . Consider  $\Omega'/G_{\Delta} \subset I^+(\Delta)/G_{\Delta} \simeq \mathbb{E}_0^{1,2}$  and use the cylindrical coordinates of  $\mathbb{E}_0^{1,2}$ . Then, the surface  $\tilde{\Sigma}/G_{\Delta}$  is a Cauchy-surface of  $\Omega/G_{\Delta}$  and intersects every vertical line  $\{r = R, \theta = \theta_0\}$ . We can use Lemma 53 in [Bru16] to extend  $\tilde{\Sigma}/G_{\Delta} \setminus \{r < R\}$  to a metrically complete Cauchy-surface of  $\Omega'/G_{\Delta}$  we call  $\Sigma$ . The lift  $\tilde{\Sigma}$  of  $\Sigma$  is a metrically complete Cauchy-surface of  $\Omega'$ . Then  $\Omega'/G$  is Cauchy-complete and we have a natural injective  $\mathbb{E}^{1,2}$ -morphism  $\Omega/G \rightarrow \Omega'/G$ . Since  $\Omega/G$  is absolutely maximal, this map is onto and  $\Omega' = \Omega$ .

Finally,

$$\text{Fix}(\psi) \cap \partial\Omega = \text{Fix}(\psi) \cap \partial\Omega' \neq \emptyset.$$

□

**Corollary 3.25.** *The maximal BTZ-extension of an absolutely maximal, Cauchy-complete,  $\mathbb{E}^{1,2}$ -manifold of admissible holonomy is Cauchy-compact.*

*Proof.* The holonomy of  $M$  is admissible, in particular,  $M$  is homeomorphic to  $\Sigma^* \times \mathbb{R}$  for some compact surface  $\Sigma$  and finite number of punctures  $S$ . Moreover, the holonomy of a peripheral loop around a puncture in  $S$ , is parabolic and, from Proposition 3.22, admits an open half-line of fixed point in the boundary of  $\Omega$ . Then, by Theorem I, the maximal BTZ-extension of  $M$  is  $\tilde{\Omega}(G)/G$ , which is homeomorphic to  $\Sigma \times \mathbb{R}$  and thus Cauchy-compact.  $\square$

### 3.2 Construction of spacetimes of given admissible holonomy

Our goal is to construct a globally hyperbolic Cauchy-complete spacetime of given admissible holonomy.

**Proposition 3.26.** *Let  $\Sigma$  be a compact surface of genus  $g$  and let  $S$  be a finite subset of cardinal  $s > 0$  such that  $2g - 2 + s > 0$ . Let  $\Gamma := \pi_1(\Sigma^*)$ .*

*The map*

$$\text{dsusp}_{\mathbb{H}^2} : \begin{array}{ccc} T\text{Teich}_{g,s} & \longrightarrow & \mathcal{M}_{g,s}(\mathbb{E}^{1,2}) \\ \rho & \longmapsto & \Omega(\rho)/\rho \end{array}$$

*with  $\Omega(\rho)$  defined in Proposition 2.13, is well defined and inverse to the holonomy map.*

The main point is to prove that  $\Omega(\rho)$  is non empty. Let  $\Sigma$  be a compact surface,  $S$  a finite subset of  $\Sigma$ ,  $\Gamma := \pi_1(\Sigma^*)$  and let  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  a marked admissible representation. We assume that the linear part  $\rho_L$  of  $\rho$  is discrete and faithful and aim to construct a spacetime of holonomy  $\rho$ .

We use divergent direction fields to define a locally injective continuous  $\Gamma$ -invariant map  $\mathbb{H}^2 \times \mathbb{R}_+^* \rightarrow \mathbb{E}^{1,2}$  where  $\Gamma$  acts on trivially on  $\mathbb{R}_+^*$ , via  $\rho_L$  on  $\mathbb{H}^2$  and via  $\rho$  on  $\mathbb{E}^{1,2}$ . The image of  $\mathbb{H}^2 \times \{t\}$  will be  $\Gamma$ -invariant acausal and metrically complete surfaces in  $\mathbb{E}^{1,2}$ . This procedure is taken from a still unpublished work of Barbot and Meusburger [BC] on construction of spacetimes with particles with spind.

**Definition 3.27** (Divergent direction field). *Let  $X \subset \mathbb{H}^2$  be any subset of  $\mathbb{H}^2$ .*

- *A direction field on  $X$  is a map  $f : X \rightarrow \mathbb{E}^{1,2}$ .*
- *A direction field on  $X$  is divergent if for all  $x, y \in \mathbb{H}^2$*

$$\langle f(x) - f(y) | x + \langle x | y \rangle y \rangle \geq 0$$

- *A direction field on  $X$  is locally divergent if for all  $p \in \mathbb{H}^2$ , there exists an open neighborhood  $\mathcal{U}$  around  $p$  such that  $f|_{\mathcal{U}}$  is divergent.*

**Definition 3.28.** *A direction field  $f$  on  $\mathbb{H}^2$  is  $\rho$ -equivariant if for all  $p \in \mathbb{H}^2$  and all  $\gamma \in \Gamma$ ,*

$$f(\rho_L(\gamma)p) = \rho(\gamma)f(p).$$

**Definition 3.29.** *Let  $f$  be a direction field. Define the map*

$$\mathcal{D}_f : \begin{array}{ccc} \mathbb{H}^2 \times \mathbb{R}_+^* & \longrightarrow & \mathbb{E}^{1,2} \\ (x, t) & \longmapsto & f(x) + tx \end{array}$$

**Remark 3.30.** *Let  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  a morphism. If  $f$  is a locally divergent and  $\rho$ -equivariant direction field then there exists a unique  $\mathbb{E}^{1,2}$ -structure on  $\mathbb{H}^2/\Gamma \times \mathbb{R}_+^*$  such that  $\mathcal{D}_f$  is its developing map. Furthermore, the holonomy associated to  $\mathcal{D}_f$  is  $\rho$ .*

**Lemma 3.31.** *Let  $X \subset \mathbb{H}^2$  be an open subset and let  $f : X \rightarrow \mathbb{E}^{1,2}$  be a  $\mathcal{C}^1$  direction field on  $X$ . If for all  $\xi \in T\mathbb{H}^2$ ,  $\langle df(\xi)|\xi \rangle > 0$  then  $f$  is locally divergent.*

*Proof.* Restricting to smaller open subsets, we can assume  $X$  convex. Let  $u, v$  be two element of  $X$ , and let  $\xi$  the unique element of  $T_u X$  whose image by the exponential is  $v$ . We have :

$$v = \frac{\sinh(\|\xi\|)}{\|\xi\|} \xi + \cosh(\|\xi\|)u$$

and, since  $\langle u|\xi \rangle = 0$  :

$$v + \langle u|v \rangle = \frac{\sinh(\|\xi\|)}{\|\xi\|} \xi + \cosh(\|\xi\|) - \cosh(\|\xi\|)u \quad (11)$$

$$= \frac{\sinh(\|\xi\|)}{\|\xi\|} \xi \quad (12)$$

We also have:

$$\|u - v\|^2 = \sinh(\|\xi\|)^2 - (\cosh(\|\xi\|) - 1)^2 \quad (13)$$

$$= 2 \cosh(\|\xi\|) - 2 \quad (14)$$

$$= 4 \sinh^2(\|\xi\|/2) \quad (15)$$

Since  $f$  is  $\mathcal{C}^1$ , there is a continuous map  $\varepsilon : X \times X \rightarrow \mathbb{E}^{1,2}$ , vanishing on the diagonal, such that:

$$f(v) = f(u) + df(\xi) + \|v - u\|\varepsilon(u, v)$$

Then:

$$\langle f(v) - f(u)|\langle u|v \rangle u \rangle = \left\langle f(u) + df(\xi) + \|v - u\|\varepsilon(u, v) \left| \frac{\sinh(\|\xi\|)}{\|\xi\|} \xi \right. \right\rangle \quad (16)$$

$$= \frac{\sinh(\|\xi\|)}{\|\xi\|} \langle df(\xi)|\xi \rangle + 2 \sinh(\|\xi\|/2) \langle \varepsilon(u, v)|\xi \rangle \quad (17)$$

Let  $u_0 \in X$ , let  $\varepsilon_0 = \frac{1}{2} \min_{\xi \in T_{u_0} X, \|\xi\|=1} \langle df(\xi)|\xi \rangle$ . Since  $\varepsilon$  and  $df$  are positive and continuous on  $X \times X$ , there exists a neighborhood  $U$  of  $u_0$  such that for all  $(u, v) \in \mathcal{U}^2$ ,  $\langle \varepsilon(u, v)|\xi \rangle \leq \frac{\varepsilon_0}{2} \|\xi\|$  and  $\langle df(\xi)|\xi \rangle \geq \|\xi\|^2 \varepsilon_0$ . Then for all  $(u, v) \in \mathcal{U}^2$ ,  $\langle f(v) - f(u)|\langle u|v \rangle u \rangle \geq 0$ , thus  $f$  is divergent on  $\mathcal{U}$ .  $\square$

**Proposition 3.32.** *There exists a  $\rho$ -equivariant locally divergent field  $g$  on  $\mathbb{H}^2$  such that for all  $t \in \mathbb{R}_+^*$ ,  $\mathcal{D}_g(\mathbb{H}^2, t)$  is acausal and complete.*

*Proof.* Consider  $(\gamma T_i)_{\gamma \in \Gamma, i \in [1, n]}$  the lift of an ideal geodesic triangulation of  $\mathbb{H}^2/\Gamma$ . Take a disjoint family of embedded horodisks  $(\gamma H_j)_{j \in \{1, \dots, s\}, \gamma \in \Gamma}$ , we get a non-geodesic cellulation  $(\gamma T'_i, H_j)_{i \in [1, n], j \in [1, s]}$  where  $T'_i := T_i \cup \bigcup_{j=1}^s H_j$ .

Since  $\rho$  is admissible, for every  $j \in [1, s]$ , there exists a unique  $\Delta_j$  such that all  $\gamma$  parabolic fixing  $H_j$  set-wise fixes  $\Delta_j$  point-wise. For each  $j$ , choose a point  $p_j \in \Delta_j$  and set

$$\forall x \in \gamma H_j, f(x) = \rho(\gamma)p_j.$$

Let  $\varphi : [0, 1] \rightarrow [0, 1]$ ,  $\mathcal{C}^1$ , inscreasing and such that  $\varphi(0) = \varphi'(0) = \varphi'(1) = 0$  and  $\varphi(1) = 1$  and For  $i \in I$ , the cell  $T'_i$  is a non-geodesic hexagon  $[A_k]_{k \in \mathbb{Z}/6\mathbb{Z}}$  with  $\mathcal{C}_{pw}^1$  boundary.  $f$  is defined on the horocycle part  $[A_0 A_1] \cup [A_2 A_3] \cup [A_4 A_5]$ . Extend  $f$  on  $[A_{2k+1} A_{2k+2}]$ ,  $k \in \{0, 1, 2\}$  putting

$$f(x) = \varphi \left( \frac{d_{\mathbb{H}^2}(x, A_{2k+1})}{d_{\mathbb{H}^2}(A_{2k+1}, A_{2k+2})} \right) f(A_{2k+2}) + (1 - \varphi) \left( \frac{d_{\mathbb{H}^2}(x, A_{2k+1})}{d_{\mathbb{H}^2}(A_{2k+1}, A_{2k+2})} \right) f(A_{2k+1})$$

Then extend  $f$  in a  $\mathcal{C}^1$  way to  $T'_i$  in such a way that  $d_x f \cdot h = 0$  for  $x \in \partial T'_i$  and  $h \perp \partial T'_i$ . This way,  $f$  is a  $\mathcal{C}^1$ ,  $\Gamma$ -invariant direction field on  $\mathbb{H}^2$ . Then,  $\xi \mapsto \langle df(\gamma\xi)|\gamma\xi \rangle = \langle df(\xi)|\xi \rangle$ , is  $\Gamma$ -invariant (with  $\Gamma$  acting trivially on  $\mathbb{R}$ ), homogeneous of degree 2 on each fiber and zero in every  $TH_j$ ,  $j \in [1, s]$ . Therefore, writing  $T^1(\mathbb{H}^2/\Gamma)$  the unitary fiber bundle over  $\mathbb{H}^2/\Gamma$ ,

$$\left| \begin{array}{ccc} T^1(\mathbb{H}^2/\Gamma) & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \langle df(\xi)|\xi \rangle \end{array} \right.$$

is well defined, have compact support and is thus bounded, let  $M \in \mathbb{R}$  be its minimum. The function  $g: x \mapsto f(x) + (M+1)x$  then satisfies the hypothesis of Lemma 3.31 and is thus locally divergent.

Let  $t \in \mathbb{R}_+^*$ , the quadratic form induced on the level set  $\mathcal{D}_g(\mathbb{H}^2, t)$  by metric of  $\mathbb{E}^{1,2}$  is  $q(\xi) = \langle d(g + t\text{Id})(\xi)|\xi \rangle = \langle dg(\xi)|\xi \rangle + t\|\xi\|^2 \geq (1+t)\|\xi\|^2 > 0$ . Then, this level set is a closed spacelike hypersurface of  $\mathbb{E}^{1,2}$ . By Corollary 46, chapter 14 of [O'N83],  $\mathcal{D}_g(\mathbb{H}^2, t)$  is acausal. Since  $q(\xi) \geq (1+t)\|\xi\|^2$ ,  $x \mapsto \mathcal{D}_g(x, t)$  enlarges distances, and since  $\mathbb{H}^2$  is metrically complete,  $\mathcal{D}_g(\mathbb{H}^2, t)$  is metrically complete.  $\square$

*Proof of Proposition 3.26.* Let  $g$  given by Proposition 3.32, let  $\Omega$  be the Cauchy development of  $\mathcal{D}_g(\mathbb{H}^2, 1)$  in  $\mathbb{E}^{1,2}$  and let  $M := \Omega/\Gamma$ . The spacetime  $M$  is globally hyperbolic, Cauchy-complete, Cauchy-maximal, future complete, homeomorphic to  $\mathbb{H}^2/\Gamma \times \mathbb{R}$  and its holonomy is  $\rho$ . By Proposition 2.13,  $\Omega(\rho)$  is not empty and  $\Omega(\rho)/\rho$  is the absolutely maximal extension of  $M$ .  $\square$

**Remark 3.33.** *The introduction of divergent direction field could have been avoided since we only needed a  $\Gamma$ -invariant metrically complete and acausal surface in Minkowski space. However, the procedure is way more general and allows to construct globally hyperbolic spacetimes of holonomy  $\rho$  which is may not be admissible.*

### 3.3 $\mathbb{H}^2 - \mathbb{E}_0^{1,2}$ correspondances

The last section ended with the definition of the map  $\text{dsusp}_{\mathbb{H}^2}$ . We can thus state and prove the following Theorem.

**Theorem II.** *The following maps are well defined and bijective.*

$$\begin{array}{ccccc} \text{Teich}_{g,s} & \xrightleftharpoons[\text{susp}_{\mathbb{H}^2}^{-1}]{\text{susp}_{\mathbb{H}^2}} & \mathcal{M}_{g,s}^L(\mathbb{E}^{1,2}) & \xrightleftharpoons[\text{Reg}]{\text{BTZ-ext}} & \mathcal{M}_{g,s}(\mathbb{E}_0^{1,2}) \\ T\text{Teich}_{g,s} & \xrightleftharpoons[\text{Hol}]{\text{dsusp}_{\mathbb{H}^2}} & \mathcal{M}_{g,s}(\mathbb{E}^{1,2}) & \xrightleftharpoons[\text{Reg}]{\text{BTZ-ext}} & \mathcal{M}_{g,s}(\mathbb{E}_0^{1,2}) \end{array}$$

*Proof.* Proposition 3.26 shows that the map  $\text{dsusp}_{\mathbb{H}^2}$  is bijective and the inverse of  $\text{Hol}$ . Corollary 3.25 shows that the maximal BTZ-extension of an absolutely maximal Cauchy-complete  $\mathbb{E}^{1,2}$ -manifold of admissible holonomy is Cauchy-compact. Thus the map  $\text{BTZ-ext}$  is well defined.

Consider  $M$  a Cauchy-compact globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifold, from Theorem 2.4  $M' := \text{Reg}(M)$  is Cauchy-maximal and Cauchy-complete. By Lemma 3.19,  $M$  is absolutely maximal and by Lemma 3.21  $M'$  is absolutely maximal. Assume  $M = \Sigma \times \mathbb{R}$  with  $\text{Sing}_0(M) = S \times \mathbb{R}$ , note  $\rho: \Gamma \rightarrow \text{Isom}(\mathbb{E}^{1,2})$  its holonomy and  $\Omega$  the developpement of  $M'$ . The group  $\Gamma$  acts properly discontinuously and freely on  $\Omega$  thus  $\rho(\Gamma)$  has no elliptic element. Consider a peripheral loop  $\gamma$  around some puncture in  $S$ , it can be chosen in a neighborhood chart of a BTZ point and



thus  $\rho(\gamma)$  is parabolic. Let  $\gamma \in \Gamma$  such that  $\rho(\gamma)$  is parabolic, then, as proved in Corollary 2.16,  $\rho(\gamma)$  has a line of fixed point  $\Delta$ . Since  $M'$  is absolutely maximal Lemma 3.22 shows that  $\Delta$  is adjacent to  $\Omega$ . Then,  $\Delta$  corresponds to a BTZ-line in the maximal BTZ-extension of  $\text{Reg}(M)$  which is  $M$  and  $\rho(\gamma) \in \text{Stab}(\Delta) = \rho(\langle c \rangle)$  for some  $c$  peripheral. Since  $\rho$  is faithful,  $\gamma \in \langle c \rangle$  thus  $\gamma$  is peripheral. Finally,  $\rho_L$  is admissible and from Corollary 2.16 so is  $\rho$ .

The map  $\text{Reg}$  is then well defined and since BTZ-ext and  $\text{Reg}$  are inverse of each other, both are bijective.

□

## 4 Decorated Moduli correspondances

### 4.1 Decorated Moduli spaces and Penner surface

Let  $g \in \mathbb{N}$ ,  $s \in \mathbb{N}$  and let  $\Sigma$  be a compact surface of genus  $g$ . In this section,  $(\Sigma, S)$  denote a topological surface  $\Sigma$  together with a set of  $s$  marked point  $S$ .

In [Pen87], Penner introduced a so-called Decorated Teichmüller space  $\widehat{\text{Teich}}_{g,s}$ . Penner defines it as the fiber bundle over the Teichmüller space which points are equivalence classes of marked hyperbolic surface of finite volume with a choice of an horocycle around each cusp. Penner then constructs a polyhedral surface associated to a point of his decorated Teichmüller space. The construction described in section 1-4 of [Pen87] goes as follows.

Let  $\Sigma^*$  be a hyperbolic surface of genus  $g$  with  $s$  cusps and choose an horocycle  $h_i$  around each cusp. The universal cover of  $\Sigma^*$  is identified with the hyperbolic plane embedded into Minkowski space. Notice that a future light ray from the origin  $O$  of Minkowski corresponds to a point at infinity in the boundary of  $\mathbb{H}^2$ . Then an horoball on  $\mathbb{H}^2$  centered on a point  $r \in \partial\mathbb{H}^2$  is exactly the intersection  $J^+(p) \cap \mathbb{H}^2$  for some  $p$  on the light ray corresponding to  $r$ . Thus to each  $h_i$  corresponds a unique point  $p$  on the future light cone from the origin. The idea is then to take the boundary of the closed convex hull of the points corresponding to the horocycles  $h_i$ . Penner proves that this boundary, say  $\Sigma'$ , is polyhedral in the sense that there is locally a finite number of totally geodesic 2-facets around each point of  $I^+(O)$ . He also proves the 2-facets are all spacelike and that each future timelike ray from the origin intersects  $\Sigma'$  exactly once. Moreover, the vertices are all in the future light cone  $\partial J^+(O)$ . Since  $\Sigma'$  is  $\pi_1(\Sigma^*)$ -invariant, this means that  $\Sigma'/\pi_1(\Sigma^*)$  is naturally endowed with a singular euclidean metric.

In our framework, the first step is exactly the suspension and the second consists in taking the maximal BTZ-extension and then associate to each horocycle a point of the BTZ-lines. We thus define a Lorentzian analogue of Penner decorated Teichmüller space by defining decoration of a  $\mathbb{E}_0^{1,2}$ -manifold.

**Definition 4.1** (Decoration of  $\mathbb{E}_0^{1,2}$ -manifold). *Let  $M$  be  $\mathbb{E}_0^{1,2}$ -manifold. A decoration of  $M$  is a choice of a point  $p_\Delta$  on each connected BTZ line  $\Delta$  of  $M$ .*

**Definition 4.2.** *Let  $(M_1, h_1, p_1^1, \dots, p_s^1), (M_2, h_2, p_1^2, \dots, p_s^2)$  be two decorated marked  $\mathbb{E}_0^{1,2}$ -manifolds homeomorphic to  $\Sigma \times \mathbb{R}$  and such that for  $i = 1, 2$   $\text{Sing}_0(M_i)$  has exactly  $s$  connected components decorated by the points  $(p_j^i)_{j \in [1, s]}$ .*

*They are equivalent if there exists a  $\mathbb{E}_0^{1,2}$ -isomorphism  $\varphi : M_1 \rightarrow M_2$  such that  $\varphi(p_i^1) = p_i^2$  for  $i \in [1, s]$  and such that  $h_2^{-1} \circ \varphi \circ h_1$  is homotopic to  $\text{Id}_{\Sigma \times \mathbb{R}}$ .*

**Definition 4.3** (Decorated BTZ Moduli space). *Let  $g \in \mathbb{N}$ , let  $s \in \mathbb{N}$  and let  $\Sigma$  be a compact surface of genus  $g$ .*

The decorated  $\mathbb{E}_0^{1,2}$ -moduli space  $\widetilde{\mathcal{M}}_{g,s}(\mathbb{E}_0^{1,2})$  is the set of equivalence classes of decorated marked  $\mathbb{E}_0^{1,2}$ -manifold homeomorphic to  $\Sigma \times \mathbb{R}$  and such that  $\text{Sing}_0(M)$  has exactly  $s$  connected components.

**Remark 4.4.** We can define accordingly the Decorated linear BTZ Moduli space  $\widetilde{\mathcal{M}}_{g,s}^L(\mathbb{E}_0^{1,2})$ .

**Definition 4.5.** Let  $\mathcal{H}$  be the set of horocycle of  $\mathbb{H}^2 \subset \mathbb{E}^{1,2}$ .

$$\text{dec}^{-1} : \left| \begin{array}{ccc} \partial J^+(O) & \rightarrow & \mathcal{H} \\ p & \mapsto & \partial J^+(p) \cap \mathbb{H}^2 \end{array} \right.$$

**Lemma 4.6** ([Pen87]). The map  $\text{dec}^{-1}$  is bijective and continuous.

Write  $\text{dec}$  the inverse of  $\text{dec}^{-1}$ .

**Proposition 4.7.** The following maps are bijective :

$$\begin{array}{ccc} \widetilde{\text{Teich}}_{g,s} & \xrightleftharpoons[(\text{susp}_{\mathbb{H}^2}^{-1} \circ \text{Reg}) \oplus \text{dec}^{-1}]{(\text{BTZ-ext} \circ \text{susp}_{\mathbb{H}^2}) \oplus \text{dec}} & \widetilde{\mathcal{M}}_{g,s}^L(\mathbb{E}_0^{1,2}) \\ \\ \widetilde{T\text{Teich}}_{g,s} & \xrightleftharpoons[\text{Hol} \oplus \text{dec}^{-1}]{(\text{BTZ-ext} \circ \text{dsusp}_{\mathbb{H}^2}) \oplus \text{dec}} & \widetilde{\mathcal{M}}_{g,s}(\mathbb{E}_0^{1,2}) \end{array}$$

*Proof.* It follows from Theorem II and 4.6. □

Equivalence between marked singular euclidean surfaces is defined in a similar manner as for the other equivalences introduced so far. Then, we can define the Euclidean moduli space.

**Definition 4.8.** The Euclidean moduli space  $\mathcal{M}_{g,s}(\mathbb{E}^2)$  is the space of equivalence classes of marked singular euclidean surface homeomorphic to  $\Sigma$  with exactly  $s$  conical singularities.

The second part of the construction of Penner then defines a map

$$\widetilde{\mathcal{M}}_{g,s}^L(\mathbb{E}_0^{1,2}) \xrightarrow{\mathcal{P}} \mathcal{M}_{g,s}(\mathbb{E}^2)$$

Our objectives now are to construct an inverse to the map  $\mathcal{P}$  and to extend  $\mathcal{P}$  to  $\widetilde{\mathcal{M}}_{g,s}(\mathbb{E}_0^{1,2})$ .

## 4.2 Suspension of a singular euclidean surface

The first step of our construction of the inverse of Penner map is to describe the cellulation which comes with the Penner-Epstein surface. First, Proposition 2.6 of [Pen87] shows that the developpement of a cell of the Penner surface satisfies Ptoleme equality and thus its vertices are cocyclic. Second, it shows that a Ptoleme inequality holds for every quadrilateral about an edge of the cellulation. This is a characterisation of the Delaunay cellulation of a compact singular Euclidean surface we now describe.

**Definition 4.9.** A geodesic cellulation of  $(\Sigma, S)$  is ideal if the edges start from  $S$  and ends on  $S$ .

**Definition 4.10** (Hinge). A hinge is a euclidean quadrilateral together with a diagonal.

**Definition 4.11** (Hinge about an edge). *Let  $\mathcal{T}$  be an ideal geodesic triangulation of  $(\Sigma, S)$ .*

*The hinge about an edge  $e$  of the triangulation  $\mathcal{T}$  is the hinge obtained by gluing along  $e$  the two triangles of  $\mathcal{T}$  intersecting along  $e$ . The diagonal of this hinge is the edge  $e$ .*

**Definition 4.12** (Legal edge). *Let  $\mathcal{T}$  be an ideal geodesic triangulation of  $(\Sigma, S)$*

*An edge  $e$  of  $\mathcal{T}$  is legal if the hinge  $h$  about  $e$  is not inside the circumscribed circle around one of the triangles composing  $h$ .*

**Proposition 4.13** ([CI01]). *Let  $(\Sigma, S)$  be a singular euclidean surface. There exists a unique ideal cellulation  $D$  such that every edge of  $D$  is legal. Moreover, every cell of  $D$  is a cocyclic polygon.*

The second step of our construction of the inverse of Penner map is to use Lemma 2.4 and Corollary 2.5 of [Pen87]. we rewrite them slightly to convey our needs.

**Lemma 4.14** ([Pen87]). *Let  $C$  be a euclidean cocyclic polygon. Then there exists a totally geodesic embedding of  $C$  into  $\mathbb{E}^{1,2}$  such that the vertices of  $C$  are in the future light cone  $\partial J^+(O)$ .*

*Moreover, two such totally geodesic embeddings only differ by an isometry  $\gamma \in \text{SO}_0(1, 2)$ .*

**Definition 4.15** (Suspension of a cocyclic Euclidean polygon). *Let  $C$  a cocyclic Euclidean polygon and let  $i : C \rightarrow \mathbb{E}^{1,2}$  be the a totally geodesic embedding of  $C$  such that the vertices of  $i(C)$  are in the light cone  $\partial J^+(O)$ . Write  $q(t, x, y) = -t^2 + x^2 + y^2$ .*

*The suspension of a cocyclic Euclidean polygon  $C$  is*

$$\text{susp}_{\mathbb{E}^2}(C) := (C \times \mathbb{R}_+^*, ds^2) \quad \text{with} \quad ds^2(x) = q(i(x))dt^2 + ds_C^2(x)$$

*together with the decoration  $(i(v_i))_{i \in [1, k]}$  where  $v_1, \dots, v_k$  are the vertices of  $C$ .*

**Remark 4.16.** *The suspension of  $C$  is nothing more than the metric on the cone or rays from  $O$  intersecting a totally geodesic embedding of  $C$  enscribed in the future light cone from the origin.*

**Remark 4.17.** *This suspension does not depend on the choice of the embedding since from Lemma 4.14 all of them are isometric via a global isometry of  $\mathbb{E}^{1,2}$ .*

The next step in our construction of the inverse of Penner map is to glue the suspension of each cell of the Delaunay cellulation of the singular Euclidean surface  $(\Sigma, S)$ . In the same manner as for gluing of Euclidean triangles, gluings give rise to singularities. The next section is devoted to the proof that these singularities are BTZ lines. If the reader is convinced of this fact, he may skip the following section.

### 4.3 Gluings of future lightlike Minkowski wedge

This section details general properties of gluings of Minkowski wedges (see Definition 4.18 below). The following proofs are classical in the riemannian contex and are not much more complicated in the Lorentzian context. However, the list of singularties that arise in the Lorentzian context is longer than in the Riemannian context, a complete classification with more involved properties is describred in [BBS11, BBS14]. Since we are only interested in gluings of direct future wedges, the aim of the section is Proposition 4.24 which shows such gluings only give rise to BTZ singularities.

**Definition 4.18** (Future wedge in Minkowski). *Let  $\Delta$  be a causal line in  $\mathbb{E}^{1,2}$  and let  $\Pi_1$  and  $\Pi_2$  be two non parallel half  $(1,1)$ -planes in  $\mathbb{E}^{1,2}$  such that  $\partial \Pi_1 = \partial \Pi_2 = \Delta$ . Assume  $\Pi_i \in J^+(\Delta)$ , then the convex hull of  $\Pi_1 \cup \Pi_2$  is a futur wedge in Minkowski space of axis  $\Delta$ .*

*A wedge is oriented by taking a futur vector  $u_0$  in  $\Delta$  and a vectors  $u_i \in \Pi_i, i \in \{1, 2\}$ . Its orientation is direct if the basis  $(u_0, u_1, u_2)$  is a direct in  $\mathbb{E}^{1,2}$ . A wedge directly oriented is said direct.*

**Lemma 4.19.** *For every future direct wedges  $S$  and  $S'$ , there exists  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})$  such that  $S' = \gamma S$*

*Proof.* Applying some translation, we can choose the origin of  $\mathbb{E}^{1,2}$  to be on  $\Delta$ . Then  $\Delta$  is a point on  $\partial\mathbb{H}^2$  and the intersection of  $\Pi_1$  and  $\Pi_2$  with  $\mathbb{H}_2$  are geodesics intersecting on the boundary at  $\Delta$ . Together with the other intersection point of  $\Pi_1$  and  $\Pi_2$  with  $\partial\mathbb{H}^2$  we get a triplet of points  $(\Delta, A_1, A_2)$  on  $\mathbb{H}^2$  which totally characterises  $S$ . The direct condition is equivalent to the fact that  $\Delta < A_1 < A_2 < \Delta$  for the direct orientation on  $\partial\mathbb{H}^2$ . The action of  $\text{SO}_0(1, 2)$  is 3-transitive on  $\partial\mathbb{H}^2$  thus, given two wedges  $(\Delta, A_1, A_2)$  and  $(\Delta', A'_1, A'_2)$ , then there exists  $\gamma$  sending the one to the other.  $\square$

**Definition 4.20** (Direct wedge Gluing). *Let  $S = (\Pi_1, \Pi_2)$  and  $S' = (\Pi'_1, \Pi'_2)$  be two direct future wedges of respective axis  $\Delta$  and  $\Delta'$ . A direct identification of  $S$  to  $S'$  is an isometry  $\gamma \in \mathbb{E}^{1,2}$  sending  $\Pi_2$  on  $\Pi'_1$  and  $\Delta$  on  $\Delta'$ . The gluing of  $S$  and  $S'$  along  $\gamma$  is*

$$S \oplus^\gamma S' := (S \cup S')/\gamma$$

**Definition 4.21** (Direct cyclic wedge gluing). *Let  $n \in \mathbb{N}^*$ , let  $S^{(i)} = (\Pi_1^{(i)}, \Pi_2^{(i)})$ , for  $i \in \mathbb{Z}/n\mathbb{Z}$ , be a family of direct future wedges of respective axis  $\Delta^{(i)}$  and let  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})^n$  such that for  $i \in [1, n]$ ,  $\gamma_i \Pi_2^{(i)} = \Pi_1^{(i+1)}$ . The direct gluing of  $(S, \gamma)$  is*

$$\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}}^\gamma S^{(i)} := \left( \bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} S^{(i)} \right) / \sim \quad \text{with} \quad x \sim y \Leftrightarrow \exists i \in \mathbb{Z}/n\mathbb{Z}, \gamma_i x = y \text{ or } \gamma_i y = x$$

**Lemma 4.22.** *There exists a unique  $\mathbb{E}^{1,2}$ -structure on  $(S \oplus_\gamma S') \setminus \Delta$  which extends the  $\mathbb{E}^{1,2}$ -structure on  $\text{Int}(S) \cup \text{Int}(S')$ .*

*Moreover,  $S \oplus_\gamma S'$  is isomorphic to a direct future wedge.*

*Proof.* To begin with, the identification and the wedges are direct, then the quotient only identifies point of  $\Pi_2$  to points on  $\Pi'_1$ . The natural projection  $\pi : S \cup S' \rightarrow S \oplus_\gamma S'$  restricted to the interior of  $S$  and  $S'$  is an homeomorphism onto its image and thus there is a natural  $\mathbb{E}^{1,2}$ -structure on the image of the interior of  $S$  and  $S'$ . Notice that  $S \oplus_\gamma S'$  is simply connected.

- Define :

$$\mathcal{D} : \begin{cases} S \oplus_\gamma S' & \longrightarrow \mathbb{E}^{1,2} \\ \pi(x) & \longmapsto \begin{cases} x & \text{if } x \in S' \\ \gamma x & \text{if } x \in S \end{cases} \end{cases}$$

Since the gluing is direct,  $\mathcal{D}$  is injective and thus is a local homeomorphism. The pull back of the  $\mathbb{E}^{1,2}$ -structure of  $\mathbb{E}^{1,2}$  via  $\mathcal{D}$  defines a  $\mathbb{E}^{1,2}$ -structure on  $S \oplus_\gamma S'$  extending the one on the interior of  $S$  and  $S'$ . The image of  $\mathcal{D}$  is exactly the future direct wedge  $(\Pi_1, \Pi'_2)$

- Assume there exists a  $\mathbb{E}^{1,2}$ -structure on  $S \oplus_\gamma S'$  extending the one on the interior of the wedges. Let  $\mathcal{D}'$  be a developpement of  $S \oplus_\gamma S'$  endowed with such an  $\mathbb{E}^{1,2}$ . Then the developpement of the interior of  $S$  is  $\alpha S$  for some  $\alpha \in \text{Isom}(\mathbb{E}^{1,2})$  and the developpement of  $S'$  is  $\alpha' S$  for some  $\alpha' \in \text{Isom}(\mathbb{E}^{1,2})$ . We choose  $\mathcal{D}$  such that  $\alpha' = 1$ .

Considering a curve  $c : ]-1, 1[ \rightarrow S \oplus_\gamma S'$  such that  $c(0) = \pi(x)$  for some  $x \in \Pi_2$  then we see that

$$\forall x \in \Pi_2, \quad \alpha x = \mathcal{D}'(c(0)) = \lim_{t \rightarrow 0^-} \mathcal{D}'(c(t)) = \lim_{t \rightarrow 0^+} \mathcal{D}'(c(t)) = \mathcal{D}'(c(0)) = \alpha' \gamma = \gamma x$$

thus  $\alpha = \gamma$ . This proves that  $\mathcal{D}' = \mathcal{D}$ .

□

**Lemma 4.23.** *Let  $n \in \mathbb{N}^*$ , let  $(S^{(i)})_{i \in \mathbb{Z}/n\mathbb{Z}}$  be a family of direct future wedges and let  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})^n$  be an identification.*

*Then there exists a unique  $\mathbb{E}^{1,2}$ -structure on the cyclic gluing of  $S$  along  $\gamma$  extending the  $\mathbb{E}^{1,2}$ -structure on the interior of the  $S^{(i)}$ .*

*Proof.* Proceed the same as for Lemma 4.22. □

**Proposition 4.24.** *Let  $n \in \mathbb{N}^*$ , let  $(S^{(i)})_{i \in \mathbb{Z}/n\mathbb{Z}}$  be a finite family of direct futur wedges in  $\mathbb{E}^{1,2}$  and let  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})^n$  an identification.*

*Then there exists a unique  $\mathbb{E}_0^{1,2}$ -structure on the cyclic gluing of  $S$  via  $\gamma$  extending the  $\mathbb{E}^{1,2}$ -structure of the interior of  $S^{(i)}, i \in \mathbb{Z}/n\mathbb{Z}$ . Furthermore :*

1.  $\text{Sing}_0(M) = \Delta$
2.  $\Delta$  is a BTZ-line

*Proof.* From Lemmas 4.19, 4.22 and 4.23, the problem reduces to  $n = 1$ ,  $S = (\Pi_1, \Pi_2)$  of axis  $\Delta$  and  $\gamma$  such that  $\gamma\Pi_1 = \Pi_2$  and  $\gamma$  fixes point-wise  $\Delta$ . The isometry  $\gamma$  is then parabolic. Let  $M$  be the cyclic gluing of  $S$  via  $\gamma$ , then let  $\widetilde{M} := \bigoplus_{n \in \mathbb{Z}}^{(\gamma^n)} \gamma^n S$  be the non-cyclic gluings of the iterated of  $S$  via  $\gamma$ . Let  $\mathbb{Z}$  acts via left multiplication by  $\gamma$  on  $\widetilde{M}$ . The identity on  $\bigcup_{n \in \mathbb{Z}} \gamma^n S \rightarrow \mathbb{E}^{1,2}$  quotient out to give a  $\mathbb{Z}$ -invariant map  $\pi : \widetilde{M} \rightarrow M$  and bijection  $\widetilde{\mathcal{D}} : \widetilde{M} \rightarrow J^+(\Delta)$ . Be careful that  $\pi$  is not an homeomorphism, but we can check that the pull back via  $\pi \circ \mathcal{D}^{-1}$  of the topology of  $M$  is exactly the BTZ-topology on  $J^+(\Delta)$ . Thus by quotienting out, we obtain an homeomorphism

$$J^+(\Delta)/\langle \gamma \rangle \simeq \mathbb{E}_0^{1,2} \xrightarrow{\overline{\pi \circ \mathcal{D}^{-1}}} M$$

This homeomorphism is a  $\mathbb{E}^{1,2}$ -morphism on the interior of  $\mathcal{D}(S) \subset I^+(\Delta)$ , by Lemma 4.23, the image of the  $\mathbb{E}^{1,2}$ -structure on  $\text{Reg}(\mathbb{E}_0^{1,2})$  is thus the natural  $\mathbb{E}^{1,2}$ -structure of  $M \setminus \Delta$ . Then for any  $\mathbb{E}_0^{1,2}$ -structure on  $M$ ,  $\pi \circ \mathcal{D}^{-1}$  is a  $\mathbb{E}_0^{1,2}$ -isomorphism and we can define one as the  $\mathbb{E}_0^{1,2}$ -structure image from  $\overline{\pi \circ \mathcal{D}^{-1}}$ . □

#### 4.4 Inverse of Penner map

We now conclude our construction of the Penner map.

**Lemma 4.25.** *Let  $C_1$  and  $C_2$  be two cones in  $\mathbb{E}^{1,2}$  from  $O$  of respective direct triangular basis  $[A_1 B_1 C_1]$  and  $[A_2 B_2 C_2]$ .*

*Then there exists a unique isometry  $\gamma \in \text{SO}_0(1, 2)$  such that  $\gamma A_1 = B_2$ ,  $\gamma B_1 = A_2$  and such that  $\gamma C_1$  and  $C_2$  are on different sides of the plane  $(OB_2 A_2)$ .*

*Proof.* It is a direct Corollary of Lemma 2.3 in [Pen87]. □

**Corollary 4.26.** *Let  $C = [v_1 \dots v_p]$  and  $C' = [v'_1 \dots v'_q]$  two direct Euclidean cocyclic polygons such that lengths  $v_1 v_2$  and  $v'_1 v'_2$  are equal.*

*Then, there exists a unique  $\gamma \in \text{Isom}(\mathbb{E}^{1,2})$  sending the vertex of  $\text{susp}_{\mathbb{E}^2}(C)$  on the vertex of  $\text{susp}_{\mathbb{E}^2}(C')$  and such that  $\gamma v_1 = v'_2$ ,  $\gamma v_2 = v'_1$  and such that  $C$  and  $C'$  lie on different sides of the plane  $(Ov'_2 v'_1)$ .*

**Definition 4.27** (Suspension of a Euclidean surface). *Let  $(\Sigma, S)$  be Euclidean surface singular exactly on  $S$ , let  $C^{(i)} = [v_j^{(i)} : j = 1, \dots, p_i]$  be the Delaunay cells of  $(\Sigma, S)$ .*

*Define  $\text{susp}_{\mathbb{E}^2}(\Sigma)$ , the suspension of  $\Sigma$ , as the unique gluing of the suspension of the Delaunay cells  $C^{(i)}$  which sends the decoration  $v_a^{(b)}$  on the marking  $v_c^{(d)}$  whenever  $v_a^{(b)}$  and  $v_c^{(d)}$  are equal in  $\Sigma$ . It comes with the decoration induced by the decoration of the suspension of each cell.*

**Theorem III.** *The following maps are bijective.*

$$\begin{array}{ccc} \widetilde{\text{Teich}}_{g,s} & \xrightarrow{(\text{BTZ-ext} \circ \text{susp}_{\mathbb{E}^2}) \oplus \text{dec}} & \widetilde{\mathcal{M}}_{g,s}^L(\mathbb{E}^{1,2}) \\ & \xleftarrow{(\text{susp}_{\mathbb{E}^2}^{-1} \circ \text{Reg}) \oplus \text{dec}^{-1}} & \mathcal{M}_{g,s}(\mathbb{E}^2) \end{array} \quad \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \xleftarrow{\text{susp}_{\mathbb{E}^2}} \end{array}$$

*Proof.* By the definition of the maps  $\text{susp}_{\mathbb{E}^2}$  and  $\mathcal{P}$  and since the cellulation of the Penner surface is the Delaunay cellulation,  $\text{susp}_{\mathbb{E}^2} \circ \mathcal{P}$  is the identity on  $\text{Teich}_{g,s}$ .

Let  $[(\Sigma, S)]$  be a point of  $\mathcal{M}_{g,s}(\mathbb{E}^2)$ . By construction,  $M := \text{susp}_{\mathbb{E}^2}(\Sigma)$  is a point of  $\widetilde{\mathcal{M}}_{g,s}^L(\mathbb{E}^{1,2})$ . The universal cover of  $\text{Reg}(\text{susp}_{\mathbb{E}^2}(\Sigma))$  is then  $I^+(O)$  and the associated augmented regular domain is  $I^+(O)$  together with a set of lightlike rays  $\Delta$  from  $O$ . Each of the ray corresponds to a BTZ line of  $\text{susp}_{\mathbb{E}^2}(\Sigma)$  and the decoration gives a family of points  $p_\Delta$  on each  $\Delta$ . The totally geodesic embedding of  $\widetilde{\Sigma}^*$  in  $I^+(O)$  given by the suspension is a polyhedral surface. Furthermore, every edge of the Delaunay cellulation of  $\Sigma$  is legal and thus the hinge about an edge satisfies the Ptoleme inequality. Then, from Lemma 2.6 of [Pen87], the totally geodesic embedding of  $\widetilde{\Sigma}^*$  in  $I^+(O)$  is thus a convex polyhedral surface with vertices  $(p_\Delta)_{\Delta \in \mathfrak{D}}$  with  $\mathfrak{D}$  the set of connected component of  $\widetilde{\text{Sing}}_0(M)$ . Then, the embedding of  $\widetilde{\Sigma}^*$  is the boundary of the convex hull of the  $(p_\Delta)_{\Delta \in \mathfrak{D}}$ . Finally, the Penner surface of  $M$  is exactly the totally geodesic embedding of  $\Sigma$  in  $\mathbb{E}^{1,2}$  induced by the suspension of  $\Sigma$  and thus

$$\mathcal{P}(\text{susp}_{\mathbb{E}^2}[(\Sigma, S)]) = [(\Sigma, S)].$$

□

## 4.5 Penner surface in non-linear Cauchy-compact spacetimes with BTZ

The aim of this section is to extend Theorem III to non-linear Cauchy-compact flat spacetimes with BTZ. We are not completely successful and only prove a partial result.

**Theorem IV.** *Let  $M$  be a Cauchy-compact Cauchy-maximal globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifold. Let  $(\Delta_i)_{i \in [1,s]}$  be the connected components of  $\text{Sing}_0(M)$  and let  $(\bar{p}_i)_{i \in [1,s]}$  be a family of points such that for all  $i \in [1,s]$ ,  $\bar{p}_i \in \Delta_i$ .*

*Then, there exists a unique convex polyhedral Cauchy-surface of  $M$  with vertices  $\bar{p}_1, \dots, \bar{p}_s$ .*

Let  $g \in \mathbb{N}$  and let  $s > 0$  such that  $2g - 2 + s > 0$ . Let  $[M, \bar{p}_1, \dots, \bar{p}_s]$  be a point of  $\widetilde{\mathcal{M}}_{g,s}(\mathbb{E}_0^{1,2})$ , the fundamental group of  $M$  is a free group with at least 2 generators, it is thus non-abelian. Therefore,  $M$  automatically falls into case (iv) of Theorem 2.2, in particular the linear part of the holonomy of  $M$  is faithful and discrete. Let  $G \subset \text{Isom}(\mathbb{E}^{1,2})$  be the image of the holonomy of  $M$ , let  $\widetilde{\Omega}$  be its augmented regular domain and let  $G \subset \text{Isom}(\mathbb{E}^{1,2})$  be the image of its holonomy. Let  $(\Delta_i)_{i \in [1,s]}$  be a set of representative of the half lightlike line in  $\widetilde{\text{Sing}}_0$ , let  $(p_i)_{i \in [1,s]}$  be the decoration of the  $(\Delta_i)_{i \in [1,s]}$  and let  $G_i = \text{Stab}(p_i)$ .

We still write  $L : \text{Isom}(\mathbb{E}^{1,2}) \rightarrow \text{SO}_0(1,2)$  the projection on the linear part.

**Definition 4.28.** *Define  $K(p)$ , the closure of the convex hull of the set  $\bigcup_{i=1}^s Gp_i$  in  $\widetilde{\Omega}$ .*

**Lemma 4.29.** *For all  $i \in [1,s]$ ,  $\Delta_i \cap K(p) = [p_i, +\infty[$*

*Proof.* Let  $i \in [1, s]$ .

- Assume  $s \geq 2$ , then take  $p_j \neq p_i$ . Since  $p_j \in I^+(\Delta_i)$ , there exists a unique  $q \in \Delta_i$  such that  $p_j \in \partial J^+(q)$ . Moreover,  $J^+(p_i) \cap \partial\Omega = ]p_i, +\infty[$  thus  $q$  is in the past of  $p_i$ . Since  $G_i$  fixes point-wise  $\Delta_i$ , it acts linearly on  $J^+(q)$ . From Corollary 3.11,  $Gp_j$  is discrete, thus  $G_i p_j$  is discrete. Let  $\phi \in G_i \setminus \{1\}$ , we have  $\phi^n p_j = \lambda_n u_n$  for some  $\lambda_n \rightarrow +\infty$  and  $u_n \rightarrow p_i$ . For all  $n \in \mathbb{N}$ , the segment from  $\phi^n p_j$  to  $p_i$  is a subset of  $K(p)$  and since  $K(p)$  is closed, so is  $[p_i, +\infty[$ .

Take a compact neighborhood  $\mathcal{U}$  of  $[q, p_j]$ , from Corollary 3.11 the orbits  $Gp_k$  for  $k \in [1, s]$  are discrete, thus  $\bigcup_{k=1}^s Gp_k$  is discrete and there exists only finitely many points inside  $\mathcal{U}$ . Moreover,  $\bigcup_{k=1}^s Gp_k \cap \Delta_i = p_i$ , thus  $\mathcal{U}$  can be chosen small enough so that  $\bigcup_{k=1}^s Gp_k \cap \mathcal{U} = p_i$ . Finally,  $\Delta_i \cap K(p) = [p_i, +\infty[$ .

- Assume  $s = 1$  and  $g \geq 1$ . It is the same argument as before but instead of  $p_j$  take  $\psi p_i$  where  $\psi$  is some hyperbolic isometry of  $G$ .

□

**Lemma 4.30.** *Every timelike geodesic in  $\tilde{\Omega}$  intersects  $\Sigma$ . Moreover, once in  $K(p)$ , a future timelike geodesic does not leave  $K(p)$ .*

*Proof.* Let  $q \in \tilde{\Omega}$  and let  $u \in \mathbb{H}^2$ . Applying some isometry of  $\mathbb{E}^{1,2}$  we can assume  $u = (1, 0, 0)$ . From Theorem II, the holonomy  $\rho$  of  $M$  is admissible. The orbits any point  $v \in \partial_\infty \mathbb{H}^2$  by  $L(G)$  is thus dense. Let  $u_1$  be a lightlike vector directing the BTZ line  $\Delta$ , then  $L(G)u_1$  is dense and we can find  $u_2$  and  $u_3$  such that  $u$  is in the interior of the convex hull of  $u_1, u_2, u_3$  in  $\mathbb{H}^2$ . Let  $q_1, q_2$  and  $q_3$  be the respective decoration of the BTZ lines  $\Delta_1, \Delta_2$  and  $\Delta_3$  of direction  $u_1, u_2$  and  $u_3$  respectively.

For  $T \in \mathbb{R}$ , let  $\Pi_T$  be the horizontal plane oh height  $T$  and let  $\pi$  be the vertical projection on  $\Pi_0$ . From Lemma 4.29, for  $T$  big enough  $\Pi_T \cap K(p)$  contains the points  $q_i + t_i u_i$ , for  $i \in \{1, 2, 3\}$  and for some  $t_i > T$ . Furthermore, for  $i \in \{1, 2, 3\}$  and  $t > 0$ ,  $\pi(q_i + t u_i) = \pi(q_i) + t \pi(u_i)$  and since  $u$  is in the interior of the convex hull of the  $(u_j)_{j \in \{1, 2, 3\}}$  then the convex hull of  $\pi(\Delta_i \cap \Pi_T)$  is increasing with  $T$  and their union for  $T > 0$  is  $\Pi_0$ . We deduce that for  $t$  big enough,  $q + t u \in K(p)$ . This shows that  $q + \mathbb{R}u$  intersects  $K(p)$  and that for all  $t$  such that  $q + t u \in K(p)$ , there exists  $t' > t$  such that  $q + t' u \in K(p)$ . Since  $K(p)$  is convex, the set of  $t$  such that  $q + t u \in K(p)$  is convex thus an interval and thus an interval of the form  $[t_0, +\infty[$ .

□

**Corollary 4.31.**  *$K(p)$  is futures complete :  $J^+(K(p)) = K(p)$ .*

*Proof.* From Lemma 4.30,  $I^+(K(p)) = K(p)$  but  $K(p)$  is closed and for all  $q \in \mathbb{E}^{1,2}$ , the closure of  $I^+(q)$  is  $J^+(q)$ . Then  $J^+(K(p)) = K(p)$ .

□

**Corollary 4.32.** *For all  $x \in \Omega \cap K(p)$ ,  $J^+(x) \setminus \{x\} \subset \text{Int}(K(p))$ ,*

*Proof.* Let  $x \in \Omega \cap K(p)$ , using the same argument as for Lemma 4.30, we can find three points  $q_1, q_2, q_3$  on some BTZ-line such that the vertical projection of  $x$  on the horizontal plane is in the convex hull of the projections of  $q_1, q_2$  and  $q_3$ . The facets  $[q_i q_j x]$  are spacelike and thus  $J^+(x)$  is in the interior of the future of the convex hull of  $\{q_1, q_2, q_3, x\}$ .

□

**Lemma 4.33.**  *$\partial K(p)$  is a closed achronal topological surface.*

*Proof.* For  $q \in K(p)$ , let  $f_q : \mathbb{R}^2 \rightarrow \mathbb{R}$  the function such that  $\{(x, f_q(x)) : x \in \mathbb{R}^2\} = \partial J^+(q)$ . For all  $q \in K(p)$ ,  $f_q$  is 1-lipschitz and, up to some global isometry, we can assume  $\forall q \in K(p), f_q \geq 0$ , thus  $f : x \mapsto \inf_{q \in K(p)} f_q(x)$  is well defined and 1-lipschitz. Since for all  $q \in K(p)$ ,  $J^+(q) \subset K(p)$ , then

$$\forall q \in K(p), \forall x \in \mathbb{R}^2, \quad (x, f(x)) \in K(p)$$

Since  $K(p)$  is closed then  $\forall x \in \mathbb{R}^2, (x, f(x)) \in K(p)$  and thus  $\partial_{\mathbb{E}^{1,2}} K(p) = \{(x, f(x)) : x \in \mathbb{R}^2\}$ . Finally,  $\partial_{\mathbb{E}^{1,2}} K(p)$  is the graph of a 1-lipschitz function defined over  $\mathbb{R}^2$  and thus  $\partial_{\mathbb{E}^{1,2}} K(p)$  is a closed achronal topological surface.  $\square$

$\square$

**Proposition 4.34.** *Every causal curves in  $\Omega$  intersects  $\partial K(p)$ .*

*Proof.* Define  $\Omega'$  the interior Cauchy-developpement of  $\partial K(p)$ . Note that  $\partial K(p)$  is not a lightlike plane, then from Theorem 1.5.1 p115 in [alg10],  $\Omega'$  is a non-empty regular domain which is either future-complete, past-complete or between two lightlike planes. Since  $\Omega' \subset \Omega$ , it cannot be past-complete and from Lemma 4.30  $\Omega'$  is future complete.

From Lemma 4.29,  $\cup_{i=1}^s G[p_i, +\infty[ \subset \partial K(p)$ . Therefore, the line of fixed point of the parabolic isometries of  $G$  intersects the boundary of  $\Omega'$  and thus, by Theorem I,  $\Omega'/G$  satisfies (iii) of Lemma 3.21.  $\Omega'/G$  is thus absolutely maximal and  $\Omega' = \Omega$ .  $\square$

$\square$

**Lemma 4.35.** *The intersection of the boundary of  $K(p)$  with the boundary of  $\Omega$  is the union of the future half BTZ-lines above  $\phi p_i$  for  $i \in [1, s]$  and  $\phi \in G$ .*

$$\partial K(p) \cap \partial \Omega = \bigcup_{i=1}^s G[p_i, +\infty[$$

*Proof.* Let  $q \in \partial K(p) \cap \partial \Omega$  and let  $(q_n)_{n \in \mathbb{N}}$  be a sequence of points of the convex hull  $K(p)$  such that  $q_n \xrightarrow{n \rightarrow +\infty} q$ . Write

$$\forall n \in \mathbb{N}, \quad q_n = \sum_{\phi \in G} \sum_{i=1}^s \alpha_{i,\phi}^{(n)} \phi p_i$$

There exists a past lightlike vector  $u$  such that a plane of direction orthogonal to  $u$  is a support plane of  $\Omega$  at  $q$ . Then, for all  $x \in \Omega$ ,  $\langle x|u \rangle > \langle q|u \rangle := r_0$  and for all  $n \in \mathbb{N}$  :

$$\begin{aligned} \langle q_n|u \rangle &= \sum_{\phi \in G} \sum_{i=1}^s \alpha_{i,\phi}^{(n)} \langle \phi p_i|u \rangle \\ &= \sum_{(\phi,i) \in A_n} \alpha_{i,\phi}^{(n)} \langle \phi p_i|u \rangle + \sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} \langle \phi p_i|u \rangle \quad \text{With } \begin{cases} A_n = \{(\phi, i) \mid \langle \phi p_i|u \rangle \leq r_0 + \varepsilon_n\} \\ B_n = \{(\phi, i) \mid \langle \phi p_i|u \rangle > r_0 + \varepsilon_n\} \\ \varepsilon_n \xrightarrow{n \rightarrow +\infty} 0 \end{cases} \\ &\geq \sum_{(\phi,i) \in A_n} \alpha_{i,\phi}^{(n)} r_0 + \sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} (r_0 + \varepsilon_n) \\ &\geq r_0 + \varepsilon_n \sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} \end{aligned}$$

Take  $\varepsilon_n = \sqrt{\langle q_n|u \rangle - r_0}$  so that

$$\sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} \xrightarrow{n \rightarrow +\infty} 0.$$



Choose a past timelike vector  $v$ , there exists a support plane of  $\Omega$  of direction  $v^\perp$ . Then for all  $x \in \Omega$ ,  $\langle x|v \rangle > r_1$  for some  $r_1 \in \mathbb{R}$ . For all  $R > 0$  and all  $\varepsilon > 0$ , the domain

$$H_{R,\varepsilon} := \{x \in \Omega \mid \langle x|v \rangle < R \text{ and } \langle x|u \rangle < r_0 + \varepsilon\}$$

is relatively compact. Since  $\bigcup_{i=1}^s Gp_i$  is discrete, for every  $(R, \varepsilon) \in \mathbb{R}_+^2$  only finitely many points of  $\bigcup_{i=1}^s Gp_i$  are in  $H_{R,\varepsilon}$ .

If  $u$  is not the direction of a BTZ-line, then we can choose  $R$  such that  $R > \langle q|v \rangle$  and  $\varepsilon > 0$  such that  $H_{R,\varepsilon}$  contains no decoration. If there exists a decoration point  $a$  in  $(q + u^\perp) \setminus J^-(q)$ ,  $a - q$  is spacelike, then we choose  $v$  such that  $\langle v|a - q \rangle > 0$  and  $R \in \mathbb{R}$  such that  $\langle v|a \rangle > R > \langle v|q \rangle$ . Again we can choose  $\varepsilon$  small enough so that  $H_{R,\varepsilon}$  contains no decoration point. Either way, for  $n \in \mathbb{N}$  such that  $\varepsilon_n \leq \varepsilon$  :

$$\langle q_n|v \rangle = \sum_{(\phi,i) \in A_n} \alpha_{i,\phi}^{(n)} \langle \phi p_i|v \rangle + \sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} \langle \phi p_i|v \rangle \quad (18)$$

$$\geq \sum_{(\phi,i) \in A_n} \alpha_{i,\phi}^{(n)} R + \sum_{(\phi,i) \in B_n} \alpha_{i,\phi}^{(n)} r_1 \quad (19)$$

$$\langle q|v \rangle + o(1) \geq R + o(1) \quad (20)$$

$$(21)$$

This is absurd since  $R > \langle q|v \rangle$ .

Therefore, there exist a decoration point in  $J^-(q)$  and  $q \in [p_0, +\infty[$  for some decoration point  $p_0$ . □

**Definition 4.36.** Define  $\tilde{\Sigma}$  the boundary of  $K(p)$  for the BTZ topology on  $\tilde{\Omega}$  and define  $\Sigma := \tilde{\Sigma}/G$ .

**Remark 4.37.** We could have replaced  $K(p)$  by the closure of the convex hull of the decoration points for the BTZ topology instead of the usual topology. These two topologies coincide on  $\Omega$  thus the closures are the same on  $\Omega$ . Both closure are then future complete and thus for all decoration point  $p$ , the future set  $I^+(p)$  is in both closures. Since the closure of  $I^+(p)$  is  $J^+(p)$  for both topologies, we end up having  $[p, +\infty[$  in both closures. Finally, we see that the two closures are identical.

**Lemma 4.38.** The surface  $\tilde{\Sigma}$  is exactly

$$\tilde{\Sigma} = (\Omega \cap \partial K(p)) \cup \{Gp_1, \dots, Gp_s\}.$$

Moreover,

$$\Sigma = \partial_M(K(p)/G)$$

*Proof.* To begin with, the BTZ topology is thinner than the usual topology thus  $\tilde{\Sigma} \subset \partial K(p)$ . Since the BTZ topology coincides with the usual topology on  $\Omega$  and since the decoration points are in  $K(p)$ , then  $(\Omega \cap \partial K(p)) \cup \{Gp_1, \dots, Gp_s\} \subset \tilde{\Sigma}$ . Now, let  $p_i$  be a decoration point and let  $q \in ]p_i, +\infty[$ . Any open neighborhood of  $q$  contains a set of the form  $\diamond_{q_1}^{q_2}$  with  $q_1 \in ]p_i, q[$ . Since  $\diamond_{q_1}^{q_2} \subset J^+(p_i) \subset K(p)$ , the point  $q$  admits a neighborhood included into  $K(p)$ . Then  $q$  is not in  $\tilde{\Sigma}$ . □

The projection  $\tilde{\Omega} \rightarrow M$  is open, the second point follows.

**Proposition 4.39.**  $\Sigma$  is a convex spacelike polyhedral surface with finitely many 2-facets. Moreover the 0-facets are  $\bar{p}_1, \dots, \bar{p}_s$ .

*Proof.* From Lemma 4.38,  $\Sigma$  intersects  $\text{Sing}_0(M)$  exactly at  $\{\bar{p}_1, \dots, \bar{p}_2\}$  and is the boundary of a convex future set.  $\Sigma$  is thus convex. Notice that Corollary 27 p415 of [O'N83] is valid for a  $\mathbb{E}_0^{1,2}$ -manifold and thus  $\Sigma$  is a topological surface in  $M$ .

We first write a decomposition of  $\partial K(p)$ . Let  $\Pi$  be some support plane of  $K(p)$ , from Lemma 4.30  $\Pi$  cannot contain a timelike geodesic. Therefore,  $\Pi$  is either lightlike or spacelike. From Corollary 4.32, a lightlike plane cannot intersect  $\Omega \cap K(p)$  and from Corollary 4.34, it cannot intersect  $\Omega$ . Finally, from Lemma 4.35,  $\Pi \cap K(p) = [q, +\infty[$  for some decoration point  $q$ .

Assume  $\Pi$  is a spacelike support plane at some point  $q \in \Omega \cap K(p)$ . then  $K(p) \subset J^+(\Pi)$ . If  $\Pi$  contains less than two decoration points, then it can be moved slightly to obtain a plane  $\Pi'$  such that  $\{Gp_1, \dots, Gp_s\} \subset J^+(\Pi')$  and  $q \in I^-(\Pi')$ , absurd. Therefore,  $\Pi$  contains at least an open edge  $e = ]\phi_1 p_i, \phi_2 p_j[$  for some  $\phi_1, \phi_2 \in G$  and  $i, j \in [1, s]$  and clearly,  $q \in e$ . Let  $u \in \Pi^\perp$  a unit future timelike vector,  $u$  lies in a convex subset  $\gamma$  of a geodesics of  $\mathbb{H}^2$ . From Corollary 4.32, we obtain that  $\gamma$  is compact. We see that as long as  $\Pi$  doesn't contain three decoration points,  $u$  is in the interior of  $\gamma$ . Assume  $\Pi$  contains at least 3 decoration points. By discreteness of the set of decoration points,  $\Pi$  contains a finite number of decoration points  $\{q_1, \dots, q_n\}$ ,  $n \geq 3$ . The convex hull  $C$  of  $(q_1, \dots, q_n)$  is a convex polygon in  $\Pi$ . For all edge  $f$  on the boundary of  $C$ , the set of support plane at  $f$  is parametrized by a non trivial geodesic segment in  $\mathbb{H}^2$ . Therefore,  $\Pi \cap K(p) = C$ .

Let  $\Pi$  lightlike support plane of  $\Omega$  at some decoration point  $q$ . We can slightly rotate  $\Pi$  to obtain a support spacelike plane at  $q$ . Then the set of future unit vector normal to a spacelike support plane at  $q$  is a non-empty convex subset  $H \subsetneq \mathbb{H}^2$ . A plane corresponding to a point in the boundary of  $H$  contains at least 2 decoration points and thus  $q$  is a vertex of a spacelike 1-facet or a 2-facet.

Finally, a point of  $\partial K(p)$  is either on a spacelike facet of vertices in  $\{Gp_1, \dots, Gp_2\}$  or on an infinite lightlike ray  $[q, +\infty[$ . Furthermore, every point of  $\{Gp_1, \dots, Gp_2\}$  is a vertex of some 1-facet or 2-facet. Therefore,  $\tilde{\Sigma}$  is a spacelike polyhedral surface with 0-facets  $Gp_1, \dots, Gp_2$ . And then  $\Sigma$  is a spacelike polyhedral surface with 0-facets  $\bar{p}_1, \dots, \bar{p}_2$ .

Assume there are infinitely many 2-facets, thus there is an accumulation point of 2-facets in  $\text{Reg}(\Sigma)$  and thus in  $\tilde{\Sigma} \cap \Omega$ . Let  $q$  be an accumulation point of 2-facets in  $\tilde{\Sigma} \cap \Omega$ , from the discussion above  $q$  is either on a 1-facet or a 2-facet and in both cases, admits a neighborhood in  $\Sigma$  with at most two 2-facets. This contradicts the fact that  $q$  is an accumulation point.  $\square$

**Proposition 4.40.**  $\Sigma$  is a Cauchy-surface of  $M$ .

*Proof.* Consider an inextensible future causal curve  $\bar{c} : \mathbb{R} \rightarrow M$ . From Lemma 13 in [Bru16], it decomposes into a connected BTZ part  $\Delta$  and a connected regular part  $\bar{c}^0$  such that  $\Delta$  is in the past of  $\bar{c}^0$ . Write  $c : \mathbb{R} \rightarrow \tilde{\Omega}$  a lift of  $\bar{c}$ .

- If  $\Delta \neq \emptyset$ , let  $t_0 \in \mathbb{R}$  such that  $\bar{c}(t_0) = \max(\Delta)$ . Then  $c(t_0) \in \phi[p_i, +\infty[$  for some  $i \in [1, s]$  and  $\phi \in G$ . Then,  $\phi p_i \in \Delta$  and  $c(t_1) = \phi p_i \in \tilde{\Sigma}$  for some  $t_1 \leq t_0$ .

For  $t \in ]-\infty, t_0]$ ,  $c(t)$  is on the BTZ line through  $c(t_1)$  and thus  $\forall t \in ]-\infty, t_0] \setminus \{t_1\}, c(t) \notin \tilde{\Sigma}$ . Let  $t > t_0$ , then  $c(t_1) \in \Omega \cap K(p)$ , and then from Corollary 4.32  $\forall t' > t, c(t') \notin \partial K(p)$ . Then  $\forall t > t_0, c(t) \notin \tilde{\Sigma}$ . Finally,  $\forall t \neq t_1, c(t) \notin \tilde{\Sigma}$ .

- If  $\Delta = \emptyset$  then  $c \cap \partial K(p) \subset \Omega \cap \partial K(p)$  thus  $c \cap \partial K(p) = c \cap \tilde{\Sigma}$ . From Proposition 4.34,  $c \cap \partial K(p) \neq \emptyset$ , then let  $t_0 := c^{-1}(\min(c \cap \partial K(p)))$ . From Proposition 4.34,  $\forall t > t_0, c(t) \notin \partial K(p)$  and thus  $c \cap \partial K(p) = \{c(t_0)\}$ .

Therefore,  $\Sigma$  is a Cauchy-surface of  $M$ .  $\square$

**Theorem IV.** *Let  $M$  be a Cauchy-compact Cauchy-maximal globally hyperbolic  $\mathbb{E}_0^{1,2}$ -manifold. Let  $(\Delta_i)_{i \in [1,s]}$  be the connected components of  $\text{Sing}_0(M)$  and let  $(\bar{p}_i)_{i \in [1,s]}$  be a family of points such that for all  $i \in [1,s]$ ,  $\bar{p}_i \in \Delta_i$ .*

*Then, there exists a unique convex polyhedral Cauchy-surface of  $M$  with vertices  $\bar{p}_1, \dots, \bar{p}_s$ .*

*Proof.* Consider  $\Sigma = \partial_M(\partial K(p)/G)$ . By Proposition 4.40,  $\Sigma$  is a Cauchy-surface and by Proposition 4.39,  $\Sigma$  is a convex polyhedral surface intersecting  $\Delta_i$  exactly at  $\bar{p}_i$ .

Let  $\Sigma_1$  be another convex polyhedral Cauchy-surface of vertices  $\bar{p}_1, \dots, \bar{p}_s$ . On the one hand,  $J^+(\Sigma_1)$  contains  $K(p)$  and thus  $\Sigma$  is in the future of  $\Sigma_1$ . On the other hand, consider an edge of  $\Sigma_1$ . Since the vertices of  $\Sigma_1$  are  $\bar{p}_1, \dots, \bar{p}_s$ , this edge is a geodesic segment from some  $\bar{p}_i$  to some  $\bar{p}_j$  and thus belongs to  $K(p)/G$ . Finally,  $\Sigma_1$  lies in the future of  $\Sigma$  and these two Cauchy-surfaces are thus equal. □

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